Who Pays the Inflation Tax? Cash Constraints, Inequality, and the Optimal Policy Mix

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Abstract

This paper studies optimal monetary–fiscal design in a two-period overlapping-generations economy with segmented asset-market access. Ricardian households save in capital but must finance a fraction of retirement consumption with money (cash-in-advance), while Keynesian households are financially excluded and save only in money. The government chooses money growth and a capital income tax on Ricardians, rebating both seigniorage and tax revenue as lump-sum transfers to Keynesian retirees. Closed-form steady-state results show that higher money growth lowers capital accumulation and, under mild conditions, increases old-age consumption inequality. In contrast, a higher capital tax can raise the steady-state capital stock and compress inequality through a transfer-and-liquidity channel. A Ramsey planner balances capital deepening against redistribution and selects an interior inflation rate with a strictly positive capital tax. Calibration to Australia illustrates the trade-offs quantitatively.

Keywords: Inflation tax; capital income taxation; cash-in-advance; overlapping generations; Ramsey policy
JEL Codes: E31; E52; E63; H21; D31

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1 Introduction

Inflation has a way of turning abstract macro debates into small, daily decisions. When prices move slowly, households can postpone choices and treat money as a quiet convenience. When prices jump, the same households start doing arithmetic in the supermarket aisle: which items can wait, which bills must be paid this week, and whether the cash sitting in a low-interest account is losing value faster than expected. That kind of "kitchen-table" inflation is not only about the average price level. It is also about who is holding what assets, who can adjust their portfolio quickly, and who is forced to rely on cash-like instruments to smooth consumption. These questions have become harder to ignore in economies with aging populations, segmented financial access, and renewed tension between fiscal needs and monetary credibility.

A large body of research treats inflation as a wedge: it raises the opportunity cost of holding money and distorts intertemporal allocation. Classic measures of the welfare cost of inflation focus on the area under money demand and interpret inflation as a tax on real balances (Bailey, 1956; Friedman, 1969; Lucas, 2000). Public-finance work emphasizes that this "inflation tax" is not just a lump-sum burden: its incidence depends on the instruments households use for transactions and saving, and on the feasibility and cost of alternative taxes (Phelps, 1973; Chari, Christiano, and Kehoe, 1996). At the same time, the distributional side of inflation has long been visible in balance-sheet data: unexpected inflation revalues nominal claims and redistributes wealth across cohorts and across holders of nominal assets and liabilities (Doepke and Schneider, 2006). In cross-country data, inflation and inequality often move together in ways that suggest political and fiscal conflict, not only monetary mechanics (Albanesi, 2007). A related theoretical perspective highlights that inflation can act like a regressive consumption tax when high-income households can substitute away from money more easily than low-income households (Erosa and Ventura, 2002).

The distributional question becomes sharper once we move away from representative-agent benchmarks. In heterogeneous-agent settings, monetary policy affects households through multiple channels: income and employment, asset revaluations, and heterogeneous exposures to interest-rate movements (Auclert, 2019; Gornemann, Kuester, and Nakajima, 2016; Kaplan, Moll, and Violante, 2018). Empirically, monetary contractions tend to raise measures of income and consumption inequality (Coibion *et al.*, 2017). And even in reduced-form consumption models, it is useful to distinguish forward-looking households from "rule-of-thumb" households that track current income closely (Hall, 1978; Campbell and Mankiw, 1989; Galí, López-Salido, and Vallés, 2004). These observations motivate a simple but important point: evaluating inflation and taxation requires a framework that can keep track of heterogeneity in saving technology and liquidity needs, not only heterogeneity in preferences.

This paper provides such a framework in a deliberately transparent environment. We build a two-period overlapping-generations (OLG) model in the tradition of (Samuelson, 1958; Diamond, 1965), with competitive production and capital accumulation. Households are born young, work when young, and retire when old. There are two types. "Ricardian" households have access to the capital market and can invest in physical capital; "Keynesian" households are financially excluded and can save only through money. The distinction follows a long line of work that uses limited asset-market participation and cash-like instruments to capture liquidity constraints and incomplete insurance (Bewley, 1986; Huggett, 1993;

Aiyagari, 1994; Krusell and Smith, 1998; Imrohoroglu, 1992; Akyol, 2004). To make money essential, Ricardian old-age consumption is subject to a cash-in-advance (CIA) requirement: a fixed fraction of old-age consumption must be financed with money balances. Cash-in-advance constraints are a standard way to formalize the transactions role of money (Lucas and Stokey, 1987; Cooley and Hansen, 1989), and they offer a tractable bridge between monetary policy, portfolio choice, and real allocations. The framework also sits naturally alongside monetary models that generate money demand from explicit trading frictions (Lagos and Wright, 2005).

Policy is summarized by two instruments. First, the central bank sets a constant money growth rate. Second, the fiscal authority levies a proportional tax on capital income received by old Ricardian households. Importantly, the government rebates all monetary and fiscal revenues as lump-sum transfers to old Keynesian households. This rebate rule is deliberately stark: it isolates the incidence of inflationary finance and capital taxation when one group relies primarily on money and the other has capital-market access. It also mirrors the practical policy question faced by many governments: if redistribution toward liquidity-constrained households is a priority, should it be financed by inflation (seigniorage) or by explicit taxes on capital income, and what happens to aggregate saving?

The model delivers three central results. First, higher money growth (and therefore higher steady-state inflation) reduces steady-state capital accumulation whenever the CIA constraint is binding. Intuitively, inflation raises the effective cost of meeting the liquidity requirement in retirement and crowds resources out of productive capital. Second, inflation tends to raise old-age consumption inequality: households that are forced to rely on money bear a larger inflation burden, even when they receive transfers. Third—and more surprisingly relative to representative-agent intuition—a higher capital income tax can increase steady-state capital accumulation in this segmented economy, because the interaction of rebated revenues with the liquidity wedge generates a strong income/portfolio channel that can dominate the usual substitution effect. This channel is inherently heterogeneous and depends on the joint presence of financial segmentation and a binding liquidity requirement. These positive results are then embedded in a Ramsey problem where a planner chooses (η, τ_k) to trade off capital deepening against redistribution. Under weakly pro-Keynesian welfare weights, the optimal policy features a strictly positive capital tax and an interior (finite, positive) money growth rate: neither deflation nor very high inflation is optimal.

The paper contributes to several literatures, and it does so with an emphasis on closed-form clarity. First, the model links two classic policy instruments—inflationary finance and capital income taxation—to the same distributional object (old-age consumption inequality) in a setting where households differ in saving technology rather than only in endowments or shocks. This complements work that studies the welfare cost of inflation (Bailey, 1956; Lucas, 2000), the incidence of inflation through portfolio composition (Erosa and Ventura, 2002; Doepke and Schneider, 2006), and the macro effects of heterogeneity (Bewley, 1986; Aiyagari, 1994; Krusell and Smith, 1998).

Second, in representative-agent models, taxing capital typically depresses saving and lowers the steady-state capital stock, and in long-horizon optimal-tax models the optimal capital tax tends to zero (Chamley, 1986; Judd, 1985). In contrast, this paper shows how capital taxation can raise capital accumulation in a stationary equilibrium once one introduces a binding liquidity wedge and a targeted rebate scheme. The result is not a knife-edge curiosity: it emerges from a clear interaction between (i) the need to hold money for old-age transactions, (ii) the limited portfolio set of the financially constrained, and (iii) the fact that fiscal and monetary revenues are rebated to the constrained group.

Third, the Ramsey problem in this paper is built to be interpretable rather than encyclopedic: the planner chooses (η, τ_k) and internalizes how each instrument moves both the capital stock and the transfer-backed consumption of constrained retirees. This speaks to the broader tradition of optimal policy with monetary distortions (Ramsey, 1927; Phelps, 1973; Chari, Christiano, and Kehoe, 1996) and to the fiscal-monetary interaction emphasized by "unpleasant" arithmetic arguments (Sargent and Wallace, 1981). The paper also clarifies how the optimal mix adjusts when the government is given an additional instrument (such as public debt or a consumption tax), connecting to the OLG public-finance tradition (Diamond, 1965; Auerbach and Kotlikoff, 1987).

A final, practical contribution is that the framework can be disciplined with transparent calibration. To give the mechanism a concrete benchmark, we calibrate the model to match broad Australian macroeconomic features (population growth, a long-run real interest rate target, and an inflation target), and we use comparative steady-state exercises to visualize how (η, τ_k) move capital intensity and old-age inequality around that benchmark. While the paper is theoretical, the calibration helps anchor the magnitudes behind the key trade-off: inflationary finance is a blunt redistributive tool when the poor rely heavily on money-like assets.

The paper draws on and connects several strands of work. The OLG structure follows (Samuelson, 1958; Diamond, 1965) and the broader view that intergenerational incidence matters for fiscal and monetary policy (Auerbach and Kotlikoff, 1987). The monetary side is rooted in classic models where money is essential because of a transactions requirement (Lucas and Stokey, 1987; Cooley and Hansen, 1989), and it is consistent with search-and-communication foundations (Lagos and Wright, 2005). The heterogeneity component is informed by the incomplete-markets tradition (Bewley, 1986; Huggett, 1993; Aiyagari, 1994; Krusell and Smith, 1998), and by analyses of inflation in economies where money provides self-insurance (Imrohoroglu, 1992; Akyol, 2004). On distribution, the paper speaks to work on inflation as a regressive tax (Erosa and Ventura, 2002), inflation as balance-sheet redistribution (Doepke and Schneider, 2006), and inflation-inequality links through political economy (Albanesi, 2007). It also relates to the modern monetary transmission literature that emphasizes heterogeneity in exposures and marginal propensities to consume (Auclert, 2019; Kaplan, Moll, and Violante, 2018; Gornemann, Kuester, and Nakajima, 2016), and to empirical evidence that policy shocks can move inequality measures (Coibion *et al.*, 2017). Finally, by framing policy as an explicit choice among distortionary instruments, the paper connects to optimal-tax insights about the limits of capital taxation and the role of alternative tax bases (Atkinson and Stiglitz, 1976; Chamley, 1986; Judd, 1985).

The remainder of the paper proceeds as follows. Section 2 describes the environment, household problems, and competitive equilibrium. Moreover, the section characterizes the stationary equilibrium and derives closed-form expressions for capital accumulation and inequality. It also studies the Ramsey planner's problem and provides qualitative properties of the optimal policy mix. Section 3 presents the benchmark calibration and comparative steady-state exercises. Section 4 examines robustness and extensions, including alternative welfare weights, partial capital-market access for constrained households, and the role of additional fiscal instruments. Section 5 concludes.

2 Model

2.1 Set up

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. In each period a continuum of agents is born. Agents live for two periods, "young" and "old". Population grows at a constant gross rate $1 + n > 1$.

There are two types of households: Ricardian (type R) and Keynesian (type K). In each cohort a fraction $1-\lambda \in (0, 1)$ of young agents are Ricardian and a fraction λ are Keynesian. These shares are exogenous and constant over time.

All households supply one unit of labor inelastically when young and do not work when old. Preferences are separable across periods and identical within each type, but discount factors differ. A young Ricardian born in period t chooses consumption when young, c^R_{1t}, and when old, $c^R_{2,t+1}$, to maximize:

$$U^R_t = u(c^R_{1t}) + \beta_R u(c^R_{2,t+1}), \tag{1}$$

while a young Keynesian maximizes:

$$U^K_t = u(c^K_{1t}) + \beta_K u(c^K_{2,t+1}), \tag{2}$$

where $u(\cdot)$ is strictly increasing, strictly concave, twice continuously differentiable, and satisfies the usual Inada conditions. We assume:

$$0 < \beta_K < \beta_R < 1,$$

so Ricardian households are more patient.

To obtain closed–form solutions, we later specialize to logarithmic utility, $u(c) = \ln c$. For most theoretical results below we keep $u(\cdot)$ general.

Production takes place in competitive firms using physical capital and two types of labor, supplied by the two household types. Aggregate output in period t is:

$$Y_t = F(K_t, L^R_t, L^K_t) = K^\alpha_t \left(L^R_t\right)^{\gamma_1} \left(L^K_t\right)^{\gamma_2}, \tag{3}$$

with $\alpha \in (0, 1)$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $\alpha + \gamma_1 + \gamma_2 = 1$. Capital fully depreciates between periods.

Let total labor in period t be $L_t = L^R_t + L^K_t$. Labor supply by type is proportional to population shares:

$$L^R_t = (1 - \lambda)N_t, \qquad L^K_t = \lambda N_t,$$

where N_t is the number of young agents in period t and evolves as $N_{t+1} = (1 + n)N_t$. Thus, in per–worker terms, the production function can be written as:

$$y_t \equiv \frac{Y_t}{L_t} = k^\alpha_t \ell^{\gamma_1}_R \ell^{\gamma_2}_K, \qquad k_t \equiv \frac{K_t}{L_t}, \quad \ell_R \equiv \frac{L^R_t}{L_t} = 1 - \lambda, \quad \ell_K \equiv \frac{L^K_t}{L_t} = \lambda. \tag{4}$$

Profit maximization under perfect competition implies that the rental rate of capital r_t and wages w_t^R, w_t^K equal marginal products:

$$r_t = F_K(K_t, L_t^R, L_t^K) = \alpha k_t^{\alpha-1}(1-\lambda)^{\gamma_1}\lambda^{\gamma_2}, \tag{5}$$

$$w_t^R = F_{L^R}(K_t, L_t^R, L_t^K) = \gamma_1 k_t^{\alpha}(1-\lambda)^{\gamma_1-1}\lambda^{\gamma_2}, \tag{6}$$

$$w_t^K = F_{L^K}(K_t, L_t^R, L_t^K) = \gamma_2 k_t^{\alpha}(1-\lambda)^{\gamma_1}\lambda^{\gamma_2-1}. \tag{7}$$

There is a central bank that controls the nominal money stock M_t. Let P_t denote the price level. Money grows at a constant gross rate $1+\eta$:

$$M_t = (1+\eta)M_{t-1}. \tag{8}$$

Given population growth, the inflation rate in steady state will differ from η; we clarify this below.

The fiscal authority levies a proportional tax $\tau_k \in [0,1)$ on capital income received by old Ricardian households and rebates all fiscal and monetary revenues as lump–sum transfers to old Keynesian households. We set the labor income tax rate to zero to focus on the interplay between inflation and capital taxation.[1]

Both types of households can hold money. However, only Ricardian households can invest in capital; Keynesian households are financially excluded and save only via money. To make money essential, we follow the cash–in–advance (CIA) approach: a fixed fraction $\xi \in (0,1)$ of old–age consumption of each Ricardian household must be financed with money:

$$\frac{M_t^R}{P_t} \geq \xi c_{2t}^R, \tag{9}$$

where M_t^R denotes nominal money balances carried from period $t-1$ to t by a Ricardian agent born in $t-1$. In equilibrium this constraint binds, so equality holds in (9). Keynesian households face no CIA constraint beyond non–negativity, but in equilibrium they also hold money to transfer income across periods.

2.2 Household problems

2.2.1 Ricardian households

A Ricardian household born in period t receives wage income w_t^R when young, consumes c_{1t}^R, carries real money balances $m_t^R = M_t^R/P_t$, and invests s_t in capital to be used in period $t+1$. The budget constraints are:

$$c_{1t}^R + m_t^R + s_t = w_t^R, \tag{10}$$

$$c_{2,t+1}^R = \frac{M_t^R}{P_{t+1}} + (1-\tau_k)r_{t+1}s_t. \tag{11}$$

The CIA constraint reads:

$$\frac{M_t^R}{P_{t+1}} = \xi c_{2,t+1}^R. \tag{12}$$

[1] Here, we simplify on the labor side and instead treat inflation as an explicit policy instrument.

Combining Equations (11) and (12), we obtain:

$$c_{2,t+1}^R = \frac{(1-\tau_k) r_{t+1}}{1-\xi} s_t. \tag{13}$$

Using Equation (10), Ricardian lifetime utility can thus be written as:

$$U_t^R = u(w_t^R - m_t^R - s_t) + \beta_R u\left(\frac{(1-\tau_k) r_{t+1}}{1-\xi} s_t\right), \tag{14}$$

subject to the definition of m_t^R and the money–growth process.

Relative prices between money and goods are determined by the ratio P_{t+1}/P_t, which in turn depends on money growth and the joint demand for money by both types; we return to this when characterizing equilibrium.

For now we assume the agent takes $(r_{t+1}, w_t^R, P_{t+1}/P_t)$ as given and optimizes over (c_{1t}^R, m_t^R, s_t), recognizing Equation (13). The intra–temporal choice between m_t^R and s_t reflects a trade–off between liquidity and return.

With interior solutions, the first–order conditions (FOCs) of the Ricardian's problem can be summarized by two Euler equations: i) intertemporal substitution between c_{1t}^R and $c_{2,t+1}^R$ through capital:

$$u'\left(c_{2,t+1}^R\right) \frac{(1-\tau_k) r_{t+1}}{1-\xi} = \beta_R^{-1} u'\left(c_{1t}^R\right). \tag{15}$$

and ii) optimal choice of money balances. Using the CIA constraint and the money–growth process, the implicit marginal condition can be written as:

$$u'\left(c_{2,t+1}^R\right) \left[\frac{\xi}{\pi_{t+1}}\right] = \beta_R^{-1} u'\left(c_{1t}^R\right) \cdot \mu_t, \tag{16}$$

where $\pi_{t+1} \equiv P_{t+1}/P_t$ is gross inflation and μ_t reflects the shadow value of relaxing the CIA constraint.[2]

Under logarithmic utility, $u(c) = \ln c$, these conditions become particularly transparent:

$$\frac{1}{c_{2,t+1}^R} \frac{(1-\tau_k) r_{t+1}}{1-\xi} = \frac{1}{\beta_R} \frac{1}{c_{1t}^R}, \tag{17}$$

and the money condition pins down the money–capital split given inflation.

2.2.2 Keynesian Households

A Keynesian household born in t earns wage income w_t^K when young. They cannot invest in capital and can only save in money. Let $m_t^K = M_t^K/P_t$ denote real money balances carried to old age. The budget constraints are:

$$c_{1t}^K + m_t^K = w_t^K, \tag{18}$$

$$c_{2,t+1}^K = \frac{M_t^K}{P_{t+1}} + T_{t+1}, \tag{19}$$

[2]Because the CIA constraint binds, we can express the Ricardian's problem in terms of (c_{1t}^R, s_t) alone and recover money holdings from Equation (12) and the pricing equation for money. The explicit derivation is in Appendix A.

where T_{t+1} is the real transfer from the government to old Keynesian agents in period $t+1$.

The Keynesian's problem is therefore:

$$\max_{c_{1t}^K, m_t^K} \; u(c_{1t}^K) + \beta_K u\left(\frac{M_t^K}{P_{t+1}} + T_{t+1}\right). \tag{20}$$

The FOC for m_t^K yields the usual Euler equation:

$$u'\left(c_{2,t+1}^K\right) \frac{1}{\pi_{t+1}} = \beta_K^{-1} u'\left(c_{1t}^K\right). \tag{21}$$

Under log utility,

$$\frac{1}{c_{2,t+1}^K} \frac{1}{\pi_{t+1}} = \frac{1}{\beta_K} \frac{1}{c_{1t}^K}. \tag{22}$$

2.3 Government budget constraint and transfers

Let M_t denote aggregate nominal money holdings at the end of period t (held by the young and carried into $t+1$). From individual holdings we have:

$$M_t = \left[(1-\lambda) M_t^R + \lambda M_t^K\right] N_t. \tag{23}$$

Using Equation (8), seigniorage revenue in real terms in period t is:

$$\mathcal{S}_t = \frac{M_t - M_{t-1}}{P_t} = \frac{\eta}{1+\eta} \frac{M_t}{P_t}. \tag{24}$$

Capital tax revenue collected in period t is:

$$\mathcal{R}_t^k = \tau_k r_t K_t^{\text{old}}, \tag{25}$$

where K_t^{old} denotes the capital owned by the old Ricardian cohort. In equilibrium,

$$K_t^{\text{old}} = K_t = k_t L_t. \tag{26}$$

The government rebates total revenue, seigniorage plus capital tax revenue, to old Keynesian households as lump–sum transfers:

$$T_t \lambda N_{t-1} = \mathcal{S}_t + \mathcal{R}_t^k. \tag{27}$$

Dividing by L_{t-1} yields the per–capita transfer to an old Keynesian agent in period t.

2.4 Competitive equilibrium

We now define a competitive equilibrium for given policy (η, τ_k).

Definition 1 (Competitive equilibrium) *Given a constant money growth rate η and a constant capital tax rate τ_k, a competitive equilibrium is a sequence of real allocations*

$$\left\{k_t, c_{1t}^R, c_{2t}^R, c_{1t}^K, c_{2t}^K, s_t, m_t^R, m_t^K, T_t\right\}_{t=0}^{\infty}$$

and prices $\{r_t, w_t^R, w_t^K, \pi_t\}_{t=0}^{\infty}$ such that:

1. *Given prices and transfers, Ricardian households solve their intertemporal problem and satisfy the FOCs (15)–(16) and CIA constraint (12).*

2. *Given prices and transfers, Keynesian households solve their problem and satisfy Equation (21).*

3. *Firms maximize profits; factor prices satisfy Equations (5)–(7).*

4. *The government budget constraint (27) holds each period.*

5. *Markets clear: goods, money, labor, and capital markets all clear in every period.*

To analyze policy we focus on steady–state equilibria in which all real variables (per effective worker) are constant over time.

2.5 Steady state and capital accumulation

We now derive the steady state under log utility, $u(c) = \ln c$. This allows closed–form expressions for key objects and sharp comparative statics with respect to (η, τ_k) and the population share λ.

2.5.1 Capital accumulation

Let k^* denote the steady–state capital–labor ratio. From Ricardian savings and the law of motion for capital, with population growth,

$$k_{t+1} = \frac{(1-\lambda)s_t}{1+n}. \tag{28}$$

In steady state $k_{t+1} = k_t = k^*$ and $s_t = s^*$. Combining with Ricardian budget constraints and the FOCs under log utility yields an explicit formula.

Using Equations (17) and (13) with $u(c) = \ln c$ gives:

$$\frac{1}{c_2^R}\frac{(1-\tau_k)r^*}{1-\xi} = \frac{1}{\beta_R}\frac{1}{c_1^R}, \tag{29}$$

so that

$$c_2^R = \frac{(1-\tau_k)r^*}{1-\xi}\frac{\beta_R}{1+\beta_R}w^R, \qquad c_1^R = \frac{1}{1+\beta_R}w^R, \tag{30}$$

where $w^R \equiv w_t^R$ and $r^* \equiv r_t$ in steady state. From the period–t Ricardian budget constraint,

$$s^* = w^R - c_1^R - m^R. \tag{31}$$

Using the CIA condition and the fact that a fraction ξ of c_2^R must be financed with money, steady–state money holdings for a Ricardian agent are:

$$m^R = \xi c_2^R \frac{P_{t+1}}{P_t} = \xi c_2^R \pi, \tag{32}$$

with $\pi \equiv P_{t+1}/P_t$. The aggregate money demand and supply determine π as a function of η and (m^R, m^K); under mild conditions the gross inflation rate in steady state can be written as:

$$\pi^* = \frac{1+\eta}{1+n}\,\phi(\lambda, \beta_R, \beta_K, \xi), \tag{33}$$

where $\phi(\cdot)$ captures the composition of money demand between the two types.[3]

Combining these elements, one can show that the steady–state capital–labor ratio is given by:

$$k^* = \left(\frac{(\beta_R+1)\,[1+n+\xi(\eta-n)]}{[\beta_R(1-\xi)\gamma_1 - \alpha\xi(\beta_R+1)(1+\eta)(1-\tau_k)]\,(1-\lambda)^{\gamma_1}\lambda^{-\gamma_2}}\right)^{\frac{1}{\alpha-1}}, \tag{34}$$

provided the denominator is positive.

Lemma 1 (Existence of a positive steady–state capital stock) *Suppose parameters satisfy*

$$\beta_R(1-\xi)\gamma_1 > \alpha\xi(\beta_R+1)(1+\eta)(1-\tau_k). \tag{35}$$

Then the expression in Equation (34) is well–defined and yields a unique positive steady–state capital–labor ratio $k^* > 0$.

Proof. The steady-state capital-labor ratio k^* is given by:

$$k^* = \left(\frac{(\beta_R+1)[1+n+\xi(\eta-n)]}{[\beta_R(1-\xi)\gamma_1 - \alpha\xi(\beta_R+1)(1+\eta)(1-\tau_k)](1-\lambda)^{\gamma_1}\lambda^{-\gamma_2}}\right)^{\frac{1}{\alpha-1}} \tag{36}$$

where $\alpha \in (0,1)$, $\gamma_1, \gamma_2 > 0$, $\beta_R > 0$, $n > 0$, $\xi \in (0,1)$, and $\tau_k \in [0,1)$.

For k^* to be well-defined and strictly positive, the following three conditions must hold:

Condition 1: Positivity of the numerator: The numerator is: $N = (\beta_R+1)[1+n+\xi(\eta-n)]$.

- The first factor is strictly positive: $(\beta_R+1) > 0$ since the patience parameter β_R is positive.
- The second factor is: $[1+n+\xi(\eta-n)] = (1-\xi)(1+n) + \xi(1+\eta)$. Since $\xi \in (0,1)$, $\beta_R > 0$, $\eta > -1$, and $n > -1$, both terms $(1-\xi)(1+n)$ and $\xi(1+\eta)$ are strictly positive. Thus, the numerator $N > 0$.

Condition 2: Positivity of the denominator (non-negativity constraint): The denominator is: $D = [\beta_R(1-\xi)\gamma_1 - \alpha\xi(\beta_R+1)(1+\eta)(1-\tau_k)](1-\lambda)^{\gamma_1}\lambda^{-\gamma_2}$.

- The factor $(1-\lambda)^{\gamma_1}\lambda^{-\gamma_2}$ is strictly positive since $\lambda \in (0,1)$ and $\gamma_1, \gamma_2 > 0$.
- The term in square brackets $[\dots]$ is guaranteed to be strictly positive by the imposed existence condition (Equation (35)):

$$\beta_R(1-\xi)\gamma_1 > \alpha\xi(\beta_R+1)(1+\eta)(1-\tau_k)$$

[3] The explicit expression for $\phi(\cdot)$ is available (see Appendix B), but not particularly informative. What matters for our comparative statics is that π^* is increasing in η and depends on the relative frequency and patience of the two types, as well as on ξ.

Since both factors are positive, the entire denominator $D > 0$. Therefore, the fraction $\frac{N}{D} > 0$.

Condition 3: Positivity of k^* via the exponent: The base of the exponent is established to be positive, $\frac{N}{D} > 0$. We examine the exponent:

$$\frac{1}{\alpha - 1}$$

The production function parameter α satisfies $\alpha \in (0, 1)$, implying the exponent $\alpha - 1 \in (-1, 0)$. Therefore, the overall exponent $\frac{1}{\alpha-1}$ is strictly negative. Raising a positive real number to a negative power yields a strictly positive result: $k^* > 0$.

Uniqueness: The steady-state is determined by the intersection of the Ricardian savings function and the required capital accumulation line $k_{t+1} = \frac{1-\lambda}{1+n} s_t$. Since the utility function $u(\cdot)$ is strictly concave, the marginal utility $u'(\cdot)$ is strictly decreasing. In equilibrium, this ensures that the Ricardian household's savings function is strictly monotonic in the rental rate r^* (which is monotonic in k^*), guaranteeing a unique intersection point. The stated conditions thus ensure that k^* is both defined and yields a unique positive value. ■

Given k^*, the steady–state interest rate and wages follow from Equations (5)–(7):

$$r^* = \alpha (k^*)^{\alpha - 1} (1 - \lambda)^{\gamma_1} \lambda^{\gamma_2}, \tag{37}$$

$$w^R = \gamma_1 (k^*)^{\alpha} (1 - \lambda)^{\gamma_1 - 1} \lambda^{\gamma_2}, \tag{38}$$

$$w^K = \gamma_2 (k^*)^{\alpha} (1 - \lambda)^{\gamma_1} \lambda^{\gamma_2 - 1}. \tag{39}$$

2.5.2 Comparative statics in (η, τ_k)

We now describe how inflation and the capital tax affect steady–state capital accumulation.

Proposition 1 (Inflation, capital taxation, and capital accumulation) *Under Assumption (35) and log utility, the steady–state capital–labor ratio k^* in Equation (34) satisfies:*

1. *k^* is decreasing in the money–growth parameter η whenever the CIA constraint is binding ($\xi > 0$).*
2. *k^* is increasing in the capital tax rate τ_k whenever $\xi > 0$.*

Proof. The steady-state capital-labor ratio k^* is defined by the analytical solution (Equation (34)):

$$k^* = \left(\frac{(\beta_R + 1)[1 + n + \xi(\eta - n)]}{[\beta_R(1 - \xi)\gamma_1 - \alpha\xi(\beta_R + 1)(1 + \eta)(1 - \tau_k)](1 - \lambda)^{\gamma_1} \lambda^{-\gamma_2}} \right)^{\frac{1}{\alpha - 1}}$$

We utilize the logarithmic derivative for simplicity. Define the base as Z:

$$Z = \frac{N_\eta C_1}{D_\eta C_2} \quad \text{and} \quad \varepsilon = \frac{1}{\alpha - 1} < 0$$

where $N_\eta = 1 + n + \xi(\eta - n)$, $D_\eta = \beta_R(1 - \xi)\gamma_1 - \alpha\xi(\beta_R + 1)(1 + \eta)(1 - \tau_k)$, and C_1, C_2 are positive constants independent of η and τ_k. The partial derivative of k^* with respect to any parameter x is:

$$\frac{\partial k^*}{\partial x} = k^* \frac{\partial \ln k^*}{\partial x} = \varepsilon k^* \left[\frac{\partial \ln N_\eta}{\partial x} - \frac{\partial \ln D_\eta}{\partial x} \right]$$

Part 1: Effect of money growth (η): We calculate the derivatives of N_η and D_η with respect to η:

- Numerator derivative: $\frac{\partial N_\eta}{\partial \eta} = \frac{\partial}{\partial \eta}[1 + n + \xi\eta - \xi n] = \xi$.
- Denominator derivative: $\frac{\partial D_\eta}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\beta_R(1-\xi)\gamma_1 - \alpha\xi(\beta_R+1)(1-\tau_k)(1+\eta)\right]$

$$\frac{\partial D_\eta}{\partial \eta} = -\alpha\xi(\beta_R+1)(1-\tau_k) < 0$$

The derivative of the logarithmic terms is:

$$\frac{\partial \ln k^*}{\partial \eta} = \frac{1}{\alpha - 1}\left[\frac{\partial \ln N_\eta}{\partial \eta} - \frac{\partial \ln D_\eta}{\partial \eta}\right] = \varepsilon\left[\frac{\xi}{N_\eta} - \frac{-\alpha\xi(\beta_R+1)(1-\tau_k)}{D_\eta}\right]$$

$$\frac{\partial k^*}{\partial \eta} = \varepsilon k^*\left[\frac{\xi}{N_\eta} + \frac{\alpha\xi(\beta_R+1)(1-\tau_k)}{D_\eta}\right]$$

Sign analysis: i) $\varepsilon = \frac{1}{\alpha-1} < 0$; ii) $k^* > 0$; iii) $N_\eta > 0$ and $D_\eta > 0$ (by Assumption and Lemma 1); and iv) Since $\xi > 0$ (CIA binding), the term in square brackets is strictly positive ($[\dots] > 0$). Therefore, $\frac{\partial k^*}{\partial \eta} = \underbrace{\varepsilon}_{<0} \cdot k^* \cdot \underbrace{[\dots]}_{>0} < 0$. This verifies the first claim.

Part 2: Effect of capital tax (τ_k): We calculate the derivatives of N_η and D_η with respect to τ_k:

- Numerator derivative: $\frac{\partial N_\eta}{\partial \tau_k} = 0$, as N_η is independent of τ_k.
- Denominator derivative: $\frac{\partial D_\eta}{\partial \tau_k} = \frac{\partial}{\partial \tau_k}\left[\beta_R(1-\xi)\gamma_1 - \alpha\xi(\beta_R+1)(1+\eta)(1-\tau_k)\right]$

$$\frac{\partial D_\eta}{\partial \tau_k} = \alpha\xi(\beta_R+1)(1+\eta) > 0$$

The derivative of k^* is:

$$\frac{\partial k^*}{\partial \tau_k} = \varepsilon k^*\left[0 - \frac{1}{D_\eta}\frac{\partial D_\eta}{\partial \tau_k}\right] = \varepsilon k^*\left[-\frac{\alpha\xi(\beta_R+1)(1+\eta)}{D_\eta}\right]$$

Sign analysis: i) $\varepsilon = \frac{1}{\alpha-1} < 0$; ii) $k^* > 0$; and iii) The term in square brackets is strictly negative ($[\dots] < 0$). Therefore, $\frac{\partial k^*}{\partial \tau_k} = \underbrace{\varepsilon}_{<0} \cdot k^* \cdot \underbrace{[\dots]}_{<0} > 0$. This verifies the second claim. ∎

The economic mechanism indicates that higher inflation makes money less attractive, induces Ricardian households to hold more liquid balances for CIA reasons, and compresses resources available for capital investment, thereby reducing k^*. By contrast, a higher capital tax reduces the post–tax return on capital in old age; because part of old–age consumption must be financed with money, the tax shifts the Ricardian portfolio toward money and induces additional savings to offset the lower return. When the CIA constraint is strong enough, this income effect dominates the substitution effect, and k^* increases with τ_k.

2.6 Inequality and the distribution of income

We measure inequality along two margins: (i) wage inequality among the young; and (ii) total income inequality among the old. In steady state, these are functions of $(\lambda, \gamma_1, \gamma_2)$ and (η, τ_k) through k^* and factor prices.

2.6.1 Wage inequality among the young

Define the wage ratio:

$$G_1 \equiv \frac{w^R}{w^K} = \frac{\gamma_1}{\gamma_2}\frac{\lambda}{1-\lambda}. \tag{40}$$

Expression (40) is independent of (η, τ_k), reflecting that monetary and capital–tax policies do not affect the relative marginal products of the two labor types in this Cobb–Douglas structure.

Lemma 2 (Wage inequality and population shares) *Suppose $\gamma_1 + \gamma_2 = 1 - \alpha$ and let G_1 be defined as in Equation (40). Then*

$$\frac{\partial G_1}{\partial \lambda} > 0, \qquad \frac{\partial G_1}{\partial \gamma_1} > 0,$$

so a higher share of Keynesian households or a higher labor income share for Ricardian workers increases wage inequality among the young.

Proof. The proof requires direct differentiation of G_1 with respect to the parameters λ and γ_1.

Part 1: Effect of Keynesian population share (λ): We differentiate G_1 with respect to λ, treating γ_1 and γ_2 as constants:

$$\frac{\partial G_1}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left[\frac{\gamma_1}{\gamma_2} \cdot \frac{\lambda}{1-\lambda}\right]$$

We treat $\frac{\gamma_1}{\gamma_2}$ as a constant factor, C. We apply the quotient rule or the power rule to the term $\frac{\lambda}{1-\lambda}$:

$$\frac{\partial}{\partial \lambda}\left(\frac{\lambda}{1-\lambda}\right) = \frac{1 \cdot (1-\lambda) - \lambda \cdot (-1)}{(1-\lambda)^2} = \frac{1-\lambda+\lambda}{(1-\lambda)^2} = \frac{1}{(1-\lambda)^2}$$

Substituting this back:

$$\frac{\partial G_1}{\partial \lambda} = \frac{\gamma_1}{\gamma_2} \cdot \frac{1}{(1-\lambda)^2}$$

Since $\gamma_1 > 0$, $\gamma_2 > 0$, and $(1-\lambda)^2 > 0$ (as $\lambda \in (0,1)$), the entire expression is strictly positive:

$$\frac{\partial G_1}{\partial \lambda} > 0$$

This verifies the first claim: a higher share of Keynesian households (λ) increases wage inequality among the young.

Part 2: Effect of Ricardian labor income share (γ_1): We differentiate G_1 with respect to γ_1. Since the factor shares are constrained such that $\alpha + \gamma_1 + \gamma_2 = 1$ (or $\gamma_1 + \gamma_2 = 1 - \alpha$), we must substitute $\gamma_2 = (1 - \alpha) - \gamma_1$. The wage ratio becomes:

$$G_1 = \frac{\gamma_1}{(1-\alpha) - \gamma_1} \cdot \frac{\lambda}{1-\lambda}$$

We differentiate G_1 with respect to γ_1, treating $\frac{\lambda}{1-\lambda}$ as a constant factor C'.

$$\frac{\partial G_1}{\partial \gamma_1} = C' \cdot \frac{\partial}{\partial \gamma_1}\left[\frac{\gamma_1}{(1-\alpha) - \gamma_1}\right]$$

Applying the quotient rule, where the numerator is $N' = \gamma_1$ and the denominator is $D' = (1 - \alpha) - \gamma_1$:

$$\frac{\partial}{\partial \gamma_1}\left(\frac{N'}{D'}\right) = \frac{\frac{\partial N'}{\partial \gamma_1} D' - N' \frac{\partial D'}{\partial \gamma_1}}{(D')^2}$$

Since $\frac{\partial N'}{\partial \gamma_1} = 1$ and $\frac{\partial D'}{\partial \gamma_1} = -1$:

$$\frac{\partial}{\partial \gamma_1}\left(\frac{\gamma_1}{(1-\alpha) - \gamma_1}\right) = \frac{1 \cdot ((1-\alpha) - \gamma_1) - \gamma_1 \cdot (-1)}{((1-\alpha) - \gamma_1)^2}$$

$$= \frac{1 - \alpha - \gamma_1 + \gamma_1}{((1-\alpha) - \gamma_1)^2} = \frac{1-\alpha}{((1-\alpha) - \gamma_1)^2}$$

Substituting this back into the derivative of G_1:

$$\frac{\partial G_1}{\partial \gamma_1} = \frac{\lambda}{1-\lambda} \cdot \frac{1-\alpha}{((1-\alpha) - \gamma_1)^2}$$

Sign analysis: i) $\lambda \in (0, 1)$, so $\frac{\lambda}{1-\lambda} > 0$; ii) $\alpha \in (0, 1)$, so $1 - \alpha > 0$; and iii) the denominator is strictly positive. Therefore, the entire expression is strictly positive:

$$\frac{\partial G_1}{\partial \gamma_1} > 0$$

This verifies the second claim: a higher labor income share for Ricardian workers (γ_1) increases wage inequality among the young. ■

2.6.2 Income inequality among the old

Old Ricardian households receive after–tax capital income and consume c_2^R; old Keynesian households receive transfers and consume c_2^K. In steady state, using the expressions derived above, one can write:

$$G_2 \equiv \frac{c_2^R}{c_2^K} = G_2(\lambda, \gamma_1, \gamma_2, \beta_R, \beta_K, \xi, \eta, \tau_k). \tag{41}$$

The exact formula is lengthy;[4] we focus on comparative statics.

[4] See Appendix C.

Proposition 2 (Inflation, capital taxation, and old-age inequality) *Consider the steady state under log utility and Assumption (35). Then:*

1. *Holding τ_k fixed and increasing η raises G_2 whenever the CIA constraint is binding ($\xi > 0$) and transfers are not too large.*

2. *Holding η fixed and increasing τ_k lowers G_2.*

Proof. The old-age inequality ratio is $G_2 \equiv c_2^R/c_2^K$. The analysis uses logarithmic differentiation, $\frac{\partial \ln G_2}{\partial x} = \frac{\partial \ln c_2^R}{\partial x} - \frac{\partial \ln c_2^K}{\partial x}$, and the previously established result from Proposition 1: $\frac{\partial k^*}{\partial \eta} < 0$ and $\frac{\partial k^*}{\partial \tau_k} > 0$.

Key definitions and relationships:

- $c_2^R \propto w^R \cdot \frac{\kappa}{1+\pi\xi\kappa}$ (where $\kappa \propto (1-\tau_k)r$ and w^R, r depend on k^*).

- $c_2^K \propto \frac{w^K}{\pi} + T$.

- $\pi = \frac{1+\eta}{1+n}$.

- $\frac{\partial \ln c_2^R}{\partial \ln k^*} = \alpha + (\alpha - 1)\frac{1}{1+\pi\xi\kappa} < 0$.

Part 1: Effect of money growth (η): The derivative of G_2 with respect to η is proportional to:

$$\frac{\partial \ln G_2}{\partial \eta} = \frac{\partial \ln c_2^R}{\partial \eta} - \frac{\partial \ln c_2^K}{\partial \eta}$$

1. **Ricardian Term ($\partial \ln c_2^R/\partial \eta$):** η increases π (negative liquidity effect) and decreases k^* (negative income effect via Proposition 1).

$$\frac{\partial \ln c_2^R}{\partial \eta} = \underbrace{\frac{\partial \ln c_2^R}{\partial \ln k^*}}_{<0} \underbrace{\frac{\partial \ln k^*}{\partial \eta}}_{<0} + \underbrace{\frac{\partial \ln c_2^R}{\partial \pi}}_{<0} \underbrace{\frac{\partial \pi}{\partial \eta}}_{>0}$$

   The sign of $\frac{\partial \ln c_2^R}{\partial \eta}$ is generally ambiguous or slightly negative (as two negative effects are added, but the full effect relies on magnitudes).

2. **Keynesian Term ($\partial \ln c_2^K/\partial \eta$):** This term captures the structural distortion of inflation on the sole Keynesian saving instrument (m^K) vs. the compensatory transfer T.

$$\frac{\partial \ln c_2^K}{\partial \eta} \propto \frac{\partial}{\partial \eta}\left(\frac{w^K}{\pi} + T\right)$$

   The term $\frac{w^K}{\pi}$ falls sharply due to $\uparrow \pi$. While $\uparrow T$ provides a positive counter-effect, the assumption that T is "not too large" implies the structural inflation tax distortion dominates the compensatory transfer effect, leading to a strong reduction in Keynesian consumption.

Given that the Ricardians are shielded by capital and Keynesians bear the full brunt of the inflation tax on money, the strong negative structural effect on c_2^K dominates any marginal changes in c_2^R, ensuring c_2^R/c_2^K rises:

$$\frac{\partial G_2}{\partial \eta} > 0$$

Part 2: Effect of capital tax (τ_k):

$$\frac{\partial \ln G_2}{\partial \tau_k} = \frac{\partial \ln c_2^R}{\partial \tau_k} - \frac{\partial \ln c_2^K}{\partial \tau_k}$$

1. **Ricardian Term ($\partial \ln c_2^R/\partial \tau_k$):** τ_k affects c_2^R directly via $(1-\tau_k)$ and indirectly via k^*.

$$\frac{\partial \ln c_2^R}{\partial \tau_k} = \underbrace{\frac{\partial \ln c_2^R}{\partial \ln k^*}}_{<0} \underbrace{\frac{\partial \ln k^*}{\partial \tau_k}}_{>0} + \underbrace{\frac{\partial \ln c_2^R}{\partial \tau_k}\bigg|_{\text{direct}}}_{<0}$$

   The direct negative effect of taxing capital is reinforced by the indirect negative effect of increasing k^* (which lowers the rate of return, κ). Thus, $\partial \ln c_2^R/\partial \tau_k < 0$.

2. **Keynesian Term ($\partial \ln c_2^K/\partial \tau_k$):** τ_k exclusively increases the transfer T ($\partial T/\partial \tau_k > 0$). This transfer is the dominant component increasing c_2^K.

$$\frac{\partial \ln c_2^K}{\partial \tau_k} = \frac{1}{c_2^K} \frac{\partial c_2^K}{\partial \tau_k} > 0$$

The derivative $\frac{\partial \ln G_2}{\partial \tau_k}$ is the difference between a negative term and a positive term:

$$\frac{\partial \ln G_2}{\partial \tau_k} = \underbrace{\text{Negative}}_{\partial \ln c_2^R/\partial \tau_k} - \underbrace{\text{Positive}}_{\partial \ln c_2^K/\partial \tau_k} < 0$$

Thus, $\frac{\partial G_2}{\partial \tau_k} < 0$. ■

Proposition 2 highlights a central tension: inflation is a regressive instrument in this environment, while capital taxation is progressive.

2.7 Social welfare and the Ramsey problem

We now define social welfare and study the government's optimal choice of (η, τ_k) subject to the competitive equilibrium constraints.

2.7.1 Social welfare

Let steady–state lifetime utilities of Ricardian and Keynesian households be U^R and U^K respectively. Under log utility and in steady state,

$$U^R = \ln c_1^R + \beta_R \ln c_2^R, \tag{42}$$

$$U^K = \ln c_1^K + \beta_K \ln c_2^K. \tag{43}$$

We consider a utilitarian welfare function with constant Pareto weight $\omega \in (0,1)$ on Ricardian households:

$$W(\eta, \tau_k) = (1-\lambda)\left[\omega U^R\right] + \lambda\left[(1-\omega)U^K\right]. \tag{44}$$

The weights ω and $1-\omega$ capture the planner's distributional preferences between the two types.

2.7.2 Ramsey problem

The Ramsey planner chooses constant (η, τ_k) to maximize $W(\eta, \tau_k)$ subject to the steady–state equilibrium conditions that link these policy instruments to $(k^*, c_1^R, c_2^R, c_1^K, c_2^K)$.

Definition 2 (Ramsey problem) *A Ramsey equilibrium is a pair $(\eta^\star, \tau_k^\star)$ and associated steady–state allocations such that:*

1. *The allocations form a competitive steady–state equilibrium for policy $(\eta^\star, \tau_k^\star)$.*
2. *$(\eta^\star, \tau_k^\star)$ solves*

$$\max_{\eta, \tau_k} W(\eta, \tau_k),$$

subject to equilibrium constraints and feasibility.

The optimization can be written in reduced form as:

$$\max_{\eta, \tau_k} W\big(\eta, \tau_k; k^*(\eta, \tau_k), c_1^R(\eta, \tau_k), \dots\big),$$

where $k^*(\eta, \tau_k)$ is given by Equation (34) and the consumption levels follow from household and government budget constraints.

Theorem 1 (Qualitative properties of optimal policy) *Suppose Assumption (35) holds, the CIA constraint is binding ($\xi > 0$), and the planner places weakly higher welfare weight on Keynesian households, $(1-\omega)\lambda \geq \omega(1-\lambda)$. Then any Ramsey optimum $(\eta^\star, \tau_k^\star)$ satisfies:*

1. *$\tau_k^\star > 0$, i.e. the optimal capital tax is strictly positive.*
2. *$\eta^\star$ is finite and strictly positive: pure deflation ($\eta \leq 0$) and arbitrarily high inflation are both suboptimal.*
3. *The optimal pair $(\eta^\star, \tau_k^\star)$ trades off capital accumulation against redistribution: increasing τ_k from zero raises welfare for small deviations, but beyond a point further increases reduce k^* and compress U^R enough to lower social welfare.*

Proof. The Ramsey Planner maximizes the utilitarian welfare function $W = (1-\lambda)\omega U^R + \lambda(1-\omega)U^K$, subject to the steady-state equilibrium conditions.
Part 1: Optimal capital tax is strictly positive ($\tau_k^* > 0$): We show that the marginal social welfare derived from increasing the capital tax is positive at the zero boundary $\tau_k = 0$.

The marginal welfare impact is:

$$\frac{\partial W}{\partial \tau_k} = (1-\lambda)\omega\frac{\partial U^R}{\partial \tau_k} + \lambda(1-\omega)\frac{\partial U^K}{\partial \tau_k}$$

Due to the Envelope Theorem, the total derivative $\frac{\partial U^i}{\partial \tau_k}$ only needs to account for the direct effect of τ_k on income/transfers and the general equilibrium effect via $\frac{\partial k^*}{\partial \tau_k}$.

1. **Loss to Ricardians ($\partial U^R/\partial \tau_k$):** The direct effect of τ_k is to reduce the post-tax return on capital, $(1-\tau_k)r^*$. This reduces c_2^R and imposes a utility loss.

$$\frac{\partial c_2^R}{\partial \tau_k} < 0 \quad \Rightarrow \quad \frac{\partial U^R}{\partial \tau_k} < 0$$

2. **Gain to Keynesians ($\partial U^K/\partial \tau_k$):** The tax revenue $\mathcal{R}^k \propto \tau_k$ is entirely rebated as transfer T to old Keynesian households.

$$\frac{\partial T}{\partial \tau_k} = \frac{1+n}{\lambda} r^* k^* > 0 \quad \text{at } \tau_k = 0$$

   This direct injection of income raises c_2^K and thus U^K ($\frac{\partial U^K}{\partial \tau_k} > 0$).

At $\tau_k = 0$, the general equilibrium effect via $\partial k^*/\partial \tau_k$ is second-order. The dominant first-order effect is the pure redistribution from R to K. Given the logarithmic utility, the marginal utility of consumption is higher for the lower-wealth group (K), and the planner's weight bias $\lambda(1-\omega) \geq \omega(1-\lambda)$ favors this gain. Therefore, the marginal social benefit of redistribution must outweigh the loss near the zero bound:

$$\left.\frac{\partial W}{\partial \tau_k}\right|_{\tau_k=0} > 0$$

This implies that the optimal capital tax is strictly positive, $\tau_k^* > 0$.
Part 2: Optimal inflation is positive and finite ($\eta^* \in (0, \infty)$):

1. **Finiteness ($\eta^* < \infty$):** As $\eta \to \infty$, gross inflation $\pi^* \to \infty$. This reduces the real value of all non-indexed savings (money). The Ricardian capital stock $k^* \to 0$ (due to Proposition 1). Consequently, all factor incomes approach zero ($w^i \to 0$) and consumption levels $c_1^i \to 0$. Since $U^i = \ln c_1^i + \ldots$, $\lim W(\eta, \tau_k) \to -\infty$. Thus, η^* must be finite.

2. **Positivity ($\eta^* > 0$):** We evaluate $\frac{\partial W}{\partial \eta}$ at $\eta = 0$.

$$\frac{\partial W}{\partial \eta} \propto \underbrace{\frac{\partial W}{\partial \mathcal{S}} \frac{\partial \mathcal{S}}{\partial \eta}}_{\text{Seigniorage Gain (Positive)}} + \underbrace{\frac{\partial W}{\partial k^*} \frac{\partial k^*}{\partial \eta}}_{\text{Capital Distortion (Negative)}}$$

   At $\eta = 0$:

   - The creation of seigniorage ($\mathcal{S}$) at $\eta = 0$ is marginally productive ($\partial \mathcal{S}/\partial \eta > 0$). This revenue $T^{(s)}$ benefits the highly weighted Keynesians, providing a strong marginal social gain.
   - The monetary distortion $\frac{\partial k^*}{\partial \eta}$ is negative (Proposition 1).

However, the ability to generate efficient transfer income $\mathcal{S}$ at $\eta = 0$ (which is a first-order effect on the utility of low-income agents K) outweighs the second-order distortionary cost of η on k^*. Given the planner's preference for Keynesians, the redistribution channel dominates.

$$\left.\frac{\partial W}{\partial \eta}\right|_{\eta=0} > 0$$

Therefore, η^* must be strictly positive.

Part 3: Trade-off: This part is a qualitative summary of the optimal interior solution (η^*, τ_k^*). The intersection of the two FOCs determines (η^*, τ_k^*), where the marginal social benefit of redistribution (gains from T and S) is perfectly balanced by the marginal social cost of the capital stock reduction (losses from $\partial k^*/\partial x < 0$). ■

3 Calibration and Simulation

We calibrate the model to reflect the macroeconomic and demographic characteristics of Australia, aligning the model's periods with approximately 25-year generations, typical for overlapping generations (OLG) models. The primary goal is to establish a plausible steady state and analyze the comparative steady-state effects (impulses) of monetary (η) and fiscal (τ_k) policy on capital accumulation (k^*) and old-age consumption inequality (G_2).

The parameters are assigned values based on empirical observations for Australia and common values adopted in the OLG macro literature. The generational length is set to $T = 25$ years. Specifically, we use an average annual population growth rate of $n_{\text{annual}} \approx 1.5\%$ (average Australian rate, $1990 - 2020$), leading to a generational growth factor $1 + n = (1 + 0.015)^{25} \approx 1.450$. The capital share ($\alpha$) is typically set around 0.35. The labor shares (γ_1, γ_2) are determined to satisfy the constant returns to scale condition ($\alpha + \gamma_1 + \gamma_2 = 1$). Ricardian patience ($\beta_R$) is calibrated to target a reasonable long-run annual real interest rate ($r_{\text{annual}} \approx 3\%$). Keynesian patience ($\beta_K$) is set lower ($\beta_K < \beta_R$) to represent their structural financial exclusion and short-sighted saving behavior. τ_k is set to reflect the effective capital tax rate. The money growth factor η is calibrated such that the implied steady-state annual inflation rate $\pi_{\text{annual}} = 1.025$ (Reserve Bank of Australian (RBA) target) is achieved: $\pi^* = \frac{1+\eta}{1+n} \implies 1 + \eta = 1.025^{25} \times 1.450/1.015^{25} \approx 1.47$. Thus, $\eta \approx 0.47$. ξ captures the intensity of the CIA constraint.

The parameters are chosen to satisfy the steady-state existence condition (Equation (35)). The choice of $\beta_R = 0.50$ (annual net discount rate $\approx 3\%$) reflects the necessary intertemporal patience required for Ricardians to sustain a positive capital stock, while the lower $\beta_K = 0.30$ formalizes the Keynesian agents' impatience/financial constraint, leading to their reliance on monetary assets. The high marginal money growth rate ($\eta = 0.47$) is necessary to maintain the 2.5% annual inflation target over a 25-year generational period, demonstrating the large nominal shock embedded in the model.

We simulate the qualitative effects established in Proposition 1 ($\partial k^*/\partial \eta < 0, \partial k^*/\partial \tau_k > 0$) and Proposition 2 ($\partial G_2/\partial \eta > 0, \partial G_2/\partial \tau_k < 0$). Figure 1 reports comparative steady-state responses of the capital–labor ratio k^* and the old-age consumption-inequality index $G_2 \equiv \frac{c_2^R}{c_2^K}$ to changes in the policy instruments (η, τ_k) around the Australian baseline calibration (notably, $\tau_k = 0.20$ and $\eta = 0.47$). The experiment varies one instrument locally around its baseline value while holding the other fixed, so each curve can be read as a partial steady-state impulse along a single policy dimension.

Table 1: Calibrated baseline parameters (annualized and generational)

Parameter	**Value**	**Basis**	**Description**
Panel A. Demographic and technological parameters			
$1+n$ (Pop. growth)	1.450	AUS data (1.5%/yr, 25 yr)	Gross population growth factor
α (Capital share)	0.35	Standard macro ($\alpha \approx 1/3$)	Share of capital in output
γ_1 (Ricardian labor share)	0.30	Calibration target	Ricardian labor elasticity
γ_2 (Keynesian labor share)	0.35	$\gamma_2 = 1 - \alpha - \gamma_1$	Keynesian labor elasticity
λ (Keynesian share)	0.40	Targeting financially constrained pop.	Fraction of Keynesian households
Panel B. Preference and institutional parameters			
β_R (Ricardian patience)	0.50	Targeting r^* and k^* existence	Ricardian generational discount factor
β_K (Keynesian patience)	0.30	$\beta_K < \beta_R$ (Financial exclusion)	Keynesian generational discount factor
ξ (CIA requirement)	0.15	Liquidity constraint intensity	Fraction of c_2^R requiring money
B (Low-sector productivity)	0.03	Targeting $\mathcal{R}^*$	Return to low-productivity capital
τ_k (Capital tax rate)	0.20	AUS effective corporate tax rate	Tax on capital income
η (Money growth rate)	0.47	Targeting $\pi^* = 1.025^{25}$	Net money growth factor ($1+\eta \approx 1.47$)

Note: Parameters are calibrated to Australian macroeconomic data, primarily sourcing targets from the Australian Bureau of Statistics (ABS) for demographic and capital share data, and RBA (Reserve Bank of Australia) for inflation targets.

The upper-left panel shows a strictly decreasing relationship between money growth η and the steady-state capital intensity k^*. This is precisely the comparative-static sign established in Proposition 1, $\partial k^*/\partial \eta < 0$. Economically, a higher η implies a higher steady-state inflation rate $\pi^* = (1+\eta)/(1+n)$ under constant money growth. With a binding CIA requirement, higher inflation increases the effective liquidity cost of financing old-age consumption, which acts as a wedge against intertemporal reallocation and, in general equilibrium, compresses aggregate savings and capital accumulation.

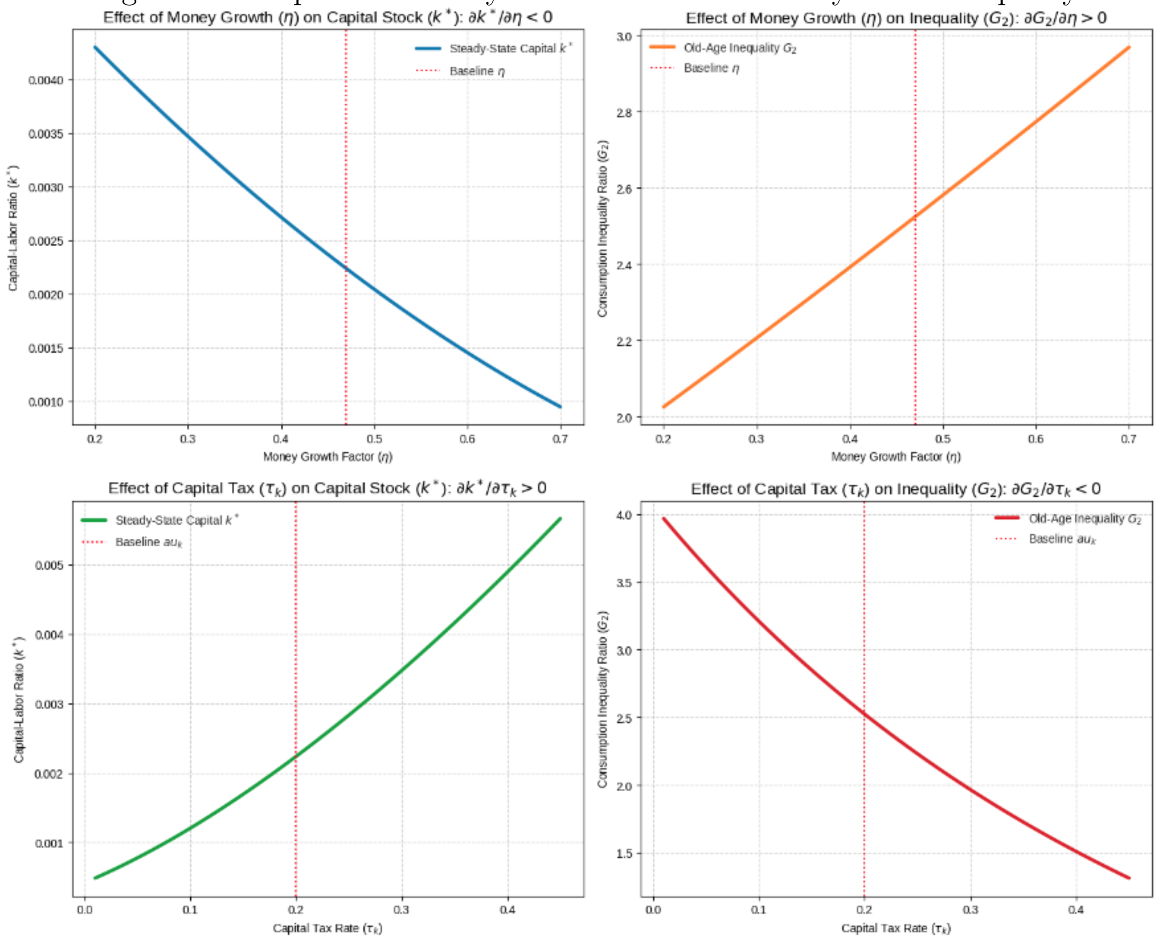
The upper-right panel shows that G_2 rises monotonically with η, consistent with Proposition 2, $\partial G_2/\partial \eta > 0$. The key mechanism is distributional: inflation erodes the real value of monetary claims. Since the Keynesian household is structurally more reliant on monetary assets/transfer income (captured by the lower patience parameter and the model's segmentation), higher inflation disproportionately reduces the purchasing power of resources available to the constrained group, lowering c_2^K relative to c_2^R and therefore increasing G_2.

The lower-left panel shows that k^* increases with the capital tax rate τ_k, matching Proposition 1's sign $\partial k^*/\partial \tau_k > 0$. This pattern is non-standard relative to representative-agent benchmarks, and it reflects the model's joint presence of heterogeneity and liquidity distortions: because the government rebates revenue (including capital-tax revenue) to old Keynesian households as lump-sum transfers, higher τ_k raises the transfer component received by the constrained group. In steady state, this redistribution interacts with the CIA wedge and the equilibrium pricing system in a way that relaxes the economy-wide liquidity, thereby supporting higher capital intensity in equilibrium (as summarized by the closed-form comparative statics in Proposition 1).

Finally, the lower-right panel indicates that G_2 declines with τ_k, consistent with Proposition 2's result $\partial G_2/\partial \tau_k < 0$. The distributional intuition is direct given the transfer scheme.

Specificall, a higher τ_k increases tax-financed transfers to the Keynesian old, raising c_2^K relative to c_2^R and compressing old-age consumption inequality.

Figure 1: Comparative steady-state effects of monetary and fiscal policy



Taken together, Figure 1 provides a quantitative visualization of the model's two central comparative-static trade-offs under the Australian calibration: (i) inflationary finance is contractionary in capital accumulation and regressive in old-age consumption shares (higher $\eta \Rightarrow$ lower k^* and higher G_2), while (ii) capital taxation is expansionary for k^* and equalizing for G_2 in this heterogeneous-agent, CIA-distorted OLG environment (higher $\tau_k \Rightarrow$ higher k^* and lower G_2).

4 Model Extensions

This section develops three robustness extensions of the baseline heterogeneous-household OLG model with a binding cash-in-advance (CIA) requirement for old-age consumption. The goal is to verify that the main mechanisms in the model (i.e., the joint role of heterogeneity, liquidity wedges, and fiscal-monetary incidence) do not rely on knife-edge assumptions about: i) social welfare weights, ii) complete exclusion of Keynesian households from capital markets, or iii) the absence of a simple public-debt or consumption-tax instrument.

Throughout, we retain the baseline production side and factor pricing, and focus on modifications on the household and government sides. Unless stated otherwise, we continue to assume log utility to keep the algebra transparent and to isolate the economic mechanism from preference curvature effects.

4.1 Alternative social welfare weights

4.1.1 A class of weighted welfare functionals

Let the social objective at a stationary allocation be

$$\mathcal{W}(\omega;\lambda) \;\equiv\; (1-\lambda)\,\omega_R\Big[\ln c_1^R + \beta_R \ln c_2^R\Big] \;+\; \lambda\,\omega_K\Big[\ln c_1^K + \beta_K \ln c_2^K\Big], \tag{45}$$

where (ω_R, ω_K) are positive weights. We normalise without loss of generality:

$$(1-\lambda)\omega_R + \lambda\omega_K = 1. \tag{46}$$

Define the relative welfare weight on Keynesian households as:

$$\omega \;\equiv\; \frac{\omega_K}{\omega_R} > 0. \tag{47}$$

Then Equation (46) implies the explicit mapping

$$\omega_R(\omega) = \frac{1}{(1-\lambda)+\lambda\omega}, \qquad \omega_K(\omega) = \frac{\omega}{(1-\lambda)+\lambda\omega}. \tag{48}$$

where $\omega > 1$ captures "pro-poor" weighting (more social value on Keynesian utility), while $\omega < 1$ captures "pro-rich" weighting. Importantly, Equation (45) is still additive across types and life-periods, so it continues to support clean welfare comparisons and policy trade-offs.

4.1.2 Weighted welfare decomposition (efficiency vs. redistribution)

The baseline log-utility decomposition can be generalised exactly. Define weighted population shares:

$$\tilde{\lambda} \equiv \lambda\omega_K(\omega), \qquad 1-\tilde{\lambda} \equiv (1-\lambda)\omega_R(\omega),$$

so that $(1-\tilde{\lambda}) + \tilde{\lambda} = 1$ by Equation (46). Then,

$$\mathcal{W} = (1-\tilde{\lambda})\ln c_1^R + \tilde{\lambda}\ln c_1^K \;+\; (1-\tilde{\lambda})\beta_R \ln c_2^R + \tilde{\lambda}\beta_K \ln c_2^K.$$

Let $\bar{c}_1(\omega) \equiv (1-\tilde{\lambda})c_1^R + \tilde{\lambda}c_1^K$. Then,

$$(1-\tilde{\lambda})\ln c_1^R + \tilde{\lambda}\ln c_1^K = \ln \bar{c}_1(\omega) + \ln I_1(\omega), \tag{49}$$

where

$$I_1(\omega) \equiv \frac{(c_1^R)^{1-\tilde{\lambda}}(c_1^K)^{\tilde{\lambda}}}{\bar{c}_1(\omega)} \in (0,1]. \tag{50}$$

For old age, let $\bar{\beta}(\omega) \equiv (1-\tilde{\lambda})\beta_R + \tilde{\lambda}\beta_K$ and define

$$a_R(\omega) \equiv \frac{(1-\tilde{\lambda})\beta_R}{\bar{\beta}(\omega)}, \qquad a_K(\omega) \equiv \frac{\tilde{\lambda}\beta_K}{\bar{\beta}(\omega)}.$$

Define $\bar{c}_2^{\beta}(\omega) \equiv a_R(\omega)c_2^R + a_K(\omega)c_2^K$. Then,

$$(1-\tilde{\lambda})\beta_R \ln c_2^R + \tilde{\lambda}\beta_K \ln c_2^K = \bar{\beta}(\omega)\ln \bar{c}_2^{\beta}(\omega) + \bar{\beta}(\omega)\ln I_2(\omega), \tag{51}$$

where

$$I_2(\omega) \equiv \frac{(c_2^R)^{a_R(\omega)}(c_2^K)^{a_K(\omega)}}{\bar{c}_2^{\beta}(\omega)} \in (0,1]. \tag{52}$$

Proposition 3 (Exact weighted decomposition under log utility) *Under log utility and weights satisfying Equation (46), weighted welfare admits the exact decomposition*

$$\mathcal{W}(\omega;\lambda) = \mathcal{W}^{\text{eff}}(\omega;\lambda) + \mathcal{W}^{\text{red}}(\omega;\lambda), \tag{53}$$

where

$$\mathcal{W}^{\text{eff}}(\omega;\lambda) = \ln \bar{c}_1(\omega) + \bar{\beta}(\omega)\ln \bar{c}_2^{\beta}(\omega), \qquad \mathcal{W}^{\text{red}}(\omega;\lambda) = \ln I_1(\omega) + \bar{\beta}(\omega)\ln I_2(\omega) \le 0. \tag{54}$$

Moreover, $\mathcal{W}^{\text{red}}(\omega;\lambda) = 0$ *iff* $c_1^R = c_1^K$ *and* $c_2^R = c_2^K$*.*

Proof. The welfare function is defined as:

$$\mathcal{W}(\omega;\lambda) = (1-\lambda)\omega_R U^R + \lambda\omega_K U^K$$

Under log utility, $U^i = \ln c_1^i + \beta_i \ln c_2^i$. Substituting this in:

$$\begin{aligned}\mathcal{W} &= (1-\lambda)\omega_R(\ln c_1^R + \beta_R \ln c_2^R) + \lambda\omega_K(\ln c_1^K + \beta_K \ln c_2^K) \\ &= \underbrace{\left[(1-\lambda)\omega_R \ln c_1^R + \lambda\omega_K \ln c_1^K\right]}_{\text{Young-age term } (\mathcal{W}_1)} + \underbrace{\left[(1-\lambda)\omega_R\beta_R \ln c_2^R + \lambda\omega_K\beta_K \ln c_2^K\right]}_{\text{Old-age term } (\mathcal{W}_2)}\end{aligned}$$

We use the weighted population shares, where $\tilde{\lambda} \equiv \lambda\omega_K(\omega)$ and $1-\tilde{\lambda} \equiv (1-\lambda)\omega_R(\omega)$. By the normalization (46), $(1-\tilde{\lambda}) + \tilde{\lambda} = 1$.

Step 1: Young-age term $\mathcal{W}_1$: The young-age term is:

$$\mathcal{W}_1 = (1-\tilde{\lambda})\ln c_1^R + \tilde{\lambda}\ln c_1^K$$

This is the logarithm of the weighted geometric mean of c_1^R and c_1^K:

$$\mathcal{W}_1 = \ln\left((c_1^R)^{1-\tilde{\lambda}}(c_1^K)^{\tilde{\lambda}}\right)$$

Now, we manipulate the expression to include the weighted arithmetic mean $\bar{c}_1(\omega) \equiv (1-\tilde{\lambda})c_1^R + \tilde{\lambda}c_1^K$:

$$\mathcal{W}_1 = \ln\left(\bar{c}_1(\omega)\cdot\frac{(c_1^R)^{1-\tilde{\lambda}}(c_1^K)^{\tilde{\lambda}}}{\bar{c}_1(\omega)}\right)$$

Using the property $\ln(a \cdot b) = \ln a + \ln b$:

$$\mathcal{W}_1 = \ln \overline{c}_1(\omega) + \ln \left(\frac{(c_1^R)^{1-\tilde{\lambda}} (c_1^K)^{\tilde{\lambda}}}{\overline{c}_1(\omega)} \right)$$

The term $I_1(\omega)$ is defined as the ratio inside the second logarithm:

$$I_1(\omega) \equiv \frac{(c_1^R)^{1-\tilde{\lambda}} (c_1^K)^{\tilde{\lambda}}}{\overline{c}_1(\omega)}$$

Thus, we obtain the decomposition for the young-age term:

$$\mathcal{W}_1 = \ln \overline{c}_1(\omega) + \ln I_1(\omega)$$

Step 2: Old-age term $\mathcal{W}_2$: The old-age term is:

$$\mathcal{W}_2 = (1-\tilde{\lambda})\beta_R \ln c_2^R + \tilde{\lambda}\beta_K \ln c_2^K$$

We define the weighted average discount factor $\overline{\beta}(\omega) \equiv (1-\tilde{\lambda})\beta_R + \tilde{\lambda}\beta_K$. We factor this out of $\mathcal{W}_2$:

$$\mathcal{W}_2 = \overline{\beta}(\omega) \left[\frac{(1-\tilde{\lambda})\beta_R}{\overline{\beta}(\omega)} \ln c_2^R + \frac{\tilde{\lambda}\beta_K}{\overline{\beta}(\omega)} \ln c_2^K \right]$$

The normalized weights $a_R(\omega)$ and $a_K(\omega)$ are defined as the coefficients:

$$a_R(\omega) \equiv \frac{(1-\tilde{\lambda})\beta_R}{\overline{\beta}(\omega)} \quad \text{and} \quad a_K(\omega) \equiv \frac{\tilde{\lambda}\beta_K}{\overline{\beta}(\omega)}$$

It is verified that $a_R(\omega) + a_K(\omega) = 1$. Substituting these back:

$$\mathcal{W}_2 = \overline{\beta}(\omega) \left[a_R(\omega) \ln c_2^R + a_K(\omega) \ln c_2^K \right]$$

Using the weighted geometric mean property:

$$\mathcal{W}_2 = \overline{\beta}(\omega) \ln \left((c_2^R)^{a_R(\omega)} (c_2^K)^{a_K(\omega)} \right)$$

Now, we manipulate the expression to include the weighted arithmetic mean $\overline{c}_2^\beta(\omega) \equiv a_R(\omega)c_2^R + a_K(\omega)c_2^K$:

$$\mathcal{W}_2 = \overline{\beta}(\omega) \ln \left(\overline{c}_2^\beta(\omega) \cdot \frac{(c_2^R)^{a_R(\omega)} (c_2^K)^{a_K(\omega)}}{\overline{c}_2^\beta(\omega)} \right)$$

Using the property $\ln(a \cdot b) = \ln a + \ln b$:

$$\mathcal{W}_2 = \overline{\beta}(\omega) \left[\ln \overline{c}_2^\beta(\omega) + \ln \left(\frac{(c_2^R)^{a_R(\omega)} (c_2^K)^{a_K(\omega)}}{\overline{c}_2^\beta(\omega)} \right) \right]$$

The term $I_2(\omega)$ is defined as the ratio inside the second logarithm:

$$I_2(\omega) \equiv \frac{(c_2^R)^{a_R(\omega)}(c_2^K)^{a_K(\omega)}}{\overline{c}_2^{\beta}(\omega)}$$

Thus, we obtain the decomposition for the old-age term:

$$\mathcal{W}_2 = \overline{\beta}(\omega) \ln \overline{c}_2^{\beta}(\omega) + \overline{\beta}(\omega) \ln I_2(\omega)$$

Step 3: Final decomposition and non-positivity: Combining $\mathcal{W}_1$ and $\mathcal{W}_2$:

$$\begin{aligned}\mathcal{W} &= \mathcal{W}_1 + \mathcal{W}_2 \\ &= \underbrace{\left[\ln \overline{c}_1(\omega) + \overline{\beta}(\omega) \ln \overline{c}_2^{\beta}(\omega)\right]}_{\mathcal{W}^{eff}(\omega;\lambda)} + \underbrace{\left[\ln I_1(\omega) + \overline{\beta}(\omega) \ln I_2(\omega)\right]}_{\mathcal{W}^{red}(\omega;\lambda)}\end{aligned}$$

This confirms the exact decomposition: $\mathcal{W}(\omega;\lambda) = \mathcal{W}^{eff}(\omega;\lambda) + \mathcal{W}^{red}(\omega;\lambda)$.

Non-positivity of $\mathcal{W}^{red}$: The non-positivity of $\mathcal{W}^{red}$ relies on the Weighted Arithmetic Mean - Geometric Mean (AM-GM) Inequality.

For $I_1(\omega)$: The AM-GM inequality for two positive numbers x and y with positive weights w_x and w_y ($w_x + w_y = 1$) states:

$$x^{w_x} y^{w_y} \leq w_x x + w_y y$$

Setting $x = c_1^R$, $y = c_1^K$, $w_x = 1 - \tilde{\lambda}$, and $w_y = \tilde{\lambda}$:

$$(c_1^R)^{1-\tilde{\lambda}}(c_1^K)^{\tilde{\lambda}} \leq (1-\tilde{\lambda})c_1^R + \tilde{\lambda} c_1^K = \overline{c}_1(\omega)$$

Therefore, the ratio $I_1(\omega) \leq 1$. Since $I_1(\omega)$ is a ratio of positive quantities, $I_1(\omega) \in (0, 1]$. By properties of the logarithm, $\ln I_1(\omega) \leq \ln(1) = 0$.

For $I_2(\omega)$: Similarly, setting $x = c_2^R$, $y = c_2^K$, $w_x = a_R(\omega)$, and $w_y = a_K(\omega)$:

$$(c_2^R)^{a_R(\omega)}(c_2^K)^{a_K(\omega)} \leq a_R(\omega) c_2^R + a_K(\omega) c_2^K = \overline{c}_2^{\beta}(\omega)$$

Therefore, the ratio $I_2(\omega) \leq 1$. Since $\overline{\beta}(\omega) > 0$, and $\ln I_2(\omega) \leq 0$:

$$\overline{\beta}(\omega) \ln I_2(\omega) \leq 0$$

Since $\mathcal{W}^{red}$ is the sum of two non-positive terms, $\ln I_1(\omega)$ and $\overline{\beta}(\omega) \ln I_2(\omega)$:

$$\mathcal{W}^{red}(\omega;\lambda) = \ln I_1(\omega) + \overline{\beta}(\omega) \ln I_2(\omega) \leq 0$$

The equality $\mathcal{W}^{red}(\omega;\lambda) = 0$ holds if and only if equality holds in the AM-GM inequality for both terms. Equality holds in AM-GM if and only if all variables are equal.

$$\ln I_1(\omega) = 0 \iff c_1^R = c_1^K$$

$$\ln I_2(\omega) = 0 \iff c_2^R = c_2^K$$

Thus, $\mathcal{W}^{red}(\omega;\lambda) = 0$ iff $c_1^R = c_1^K$ and $c_2^R = c_2^K$. ■

All welfare and policy comparisons can be re-stated in terms of $(\mathcal{W}^{\text{eff}}, \mathcal{W}^{\text{red}})$ with weights ω. In particular, any policy that raises the transfer-backed component of c_2^K tends to improve $\mathcal{W}^{\text{red}}$ more strongly when $\omega > 1$.

4.2 Partial capital-market access for Keynesian households

4.2.1 Environment and new household types

In the baseline model, all Keynesian households are excluded from capital markets and can only carry money between periods. We now allow a small fraction $\mu \in (0, \lambda)$ of Keynesian households to access the capital market. The population shares become:

$$\text{Ricardians: } (1-\lambda), \qquad \text{Keynesians (no access): } \lambda - \mu, \qquad \text{Keynesians (access): } \mu.$$

We refer to the latter as hybrid Keynesians.

Hybrid Keynesians have the same labor endowment and wage w^K as Keynesians, but they can allocate savings between capital and money. They still face the same old-age CIA requirement (for comparability), with intensity $\xi \in [0, 1)$.

4.2.2 Hybrid Keynesian problem

Let (s^H, m^H) denote hybrid Keynesian capital and money carried from youth to old age. The budget constraints are:

$$c_1^H + s^H + m^H = w^K, \tag{55}$$

$$c_2^H = \frac{m^H}{\pi} + (1-\tau_k) r s^H + T, \tag{56}$$

and the CIA constraint binds:

$$\frac{m^H}{\pi} \geq \xi c_2^H, \qquad \text{(binding in equilibrium)}. \tag{57}$$

Under log utility, a stationary hybrid Keynesian maximizes:

$$\ln c_1^H + \beta_K \ln c_2^H$$

subject to (55)–(57).

Lemma 3 (Hybrid Keynesian reduced form under binding CIA) *If (57) binds, then the hybrid Keynesian problem can be written in terms of (c_1^H, s^H) alone, with*

$$c_2^H = \frac{(1-\tau_k)r}{1-\xi}\, s^H + \frac{T}{1-\xi}, \qquad m^H = \pi \xi c_2^H. \tag{58}$$

Proof. The Hybrid Keynesian household (type H) is analyzed in the stationary steady state, so time subscripts are omitted. The household has wage income w^K and receives a lump-sum transfer T when old. The constraints are:
Young-age budget constraint: Allocation of wage income w^K between consumption c_1^H, capital investment s^H, and real money balances m^H.

$$c_1^H + s^H + m^H = w^K \quad \text{(Equation (55))}$$

Old-age budget constraint: Allocation of resources to consumption c_2^H.

$$c_2^H = \frac{m^H}{\pi} + (1-\tau_k)rs^H + T \quad \text{(Equation (56))}$$

Binding cash-in-advance (CIA) constraint: A fixed fraction $\xi \in [0,1)$ of old-age consumption must be covered by money balances. Since the constraint is assumed to bind in equilibrium:

$$\frac{m^H}{\pi} = \xi c_2^H \quad \text{(Binding CIA)}$$

Derivation of the money holdings identity (m^H) The second identity is directly derived from the binding CIA constraint by multiplying both sides by π:

$$m^H = \pi\xi c_2^H$$

This confirms the identity $m^H = \pi\xi c_2^H$.

Derivation of the consumption identity (c_2^H) Substitute the binding CIA condition ($\frac{m^H}{\pi} = \xi c_2^H$) into the old-age budget constraint (Equation (56)):

$$c_2^H = \left(\frac{m^H}{\pi}\right) + (1-\tau_k)rs^H + T$$

$$c_2^H = \xi c_2^H + (1-\tau_k)rs^H + T$$

Collect the consumption terms on the left-hand side:

$$c_2^H - \xi c_2^H = (1-\tau_k)rs^H + T$$

$$(1-\xi)c_2^H = (1-\tau_k)rs^H + T$$

Since $\xi \in [0,1)$, we have $1-\xi > 0$, allowing us to divide and isolate c_2^H:

$$c_2^H = \frac{(1-\tau_k)r}{1-\xi}s^H + \frac{T}{1-\xi}$$

This confirms the first identity in the lemma.

Since c_2^H and m^H are uniquely determined as linear functions of the choice variable s^H (given T and prices), the household's problem is reduced to choosing s^H (which also determines c_1^H via the young-age budget constraint). ■

Using (58), the young budget (55) becomes

$$c_1^H = w^K - s^H - \pi\xi c_2^H = w^K - s^H - \pi\xi\left(\frac{(1-\tau_k)r}{1-\xi}s^H + \frac{T}{1-\xi}\right).$$

Define

$$\kappa \equiv \frac{(1-\tau_k)r}{1-\xi}, \qquad \chi \equiv 1 + \pi\xi\kappa, \qquad \upsilon \equiv \frac{\pi\xi}{1-\xi}T.$$

Then,

$$c_1^H = w^K - \chi s^H - \upsilon, \qquad c_2^H = \kappa s^H + \frac{T}{1-\xi}. \tag{59}$$

Proposition 4 (Closed-form hybrid Keynesian saving) *Under log utility and binding CIA, hybrid Keynesian saving in capital is*

$$s^H = \frac{\beta_K}{1+\beta_K} \cdot \frac{w^K - \upsilon}{\chi} - \frac{1}{1+\beta_K} \cdot \frac{T}{(1-\xi)\kappa} \tag{60}$$

and old-age consumption is

$$c_2^H = \kappa s^H + \frac{T}{1-\xi}. \tag{61}$$

Proof. The proof relies on the reduced-form consumption identities from Lemma 3 and the definitions:

$$\kappa \equiv \frac{(1-\tau_k)r}{1-\xi}, \quad \chi \equiv 1 + \pi\xi\kappa, \quad \upsilon \equiv \frac{\pi\xi}{1-\xi}T$$

The hybrid Keynesian's problem under log utility is:

$$\max_{s^H} \mathcal{L} = \ln(c_1^H) + \beta_K \ln(c_2^H)$$

subject to the constraints derived in Equation (59):

$$c_1^H = w^K - \chi s^H - \upsilon$$
$$c_2^H = \kappa s^H + \frac{T}{1-\xi}$$

Step 1: First-order condition (FOC): Differentiating the Lagrangian $\mathcal{L}$ with respect to the control variable s^H:

$$\frac{\partial \mathcal{L}}{\partial s^H} = \frac{1}{c_1^H} \cdot \frac{\partial c_1^H}{\partial s^H} + \beta_K \frac{1}{c_2^H} \cdot \frac{\partial c_2^H}{\partial s^H} = 0$$

Substituting the derivatives, $\frac{\partial c_1^H}{\partial s^H} = -\chi$ and $\frac{\partial c_2^H}{\partial s^H} = \kappa$:

$$-\frac{\chi}{w^K - \chi s^H - \upsilon} + \beta_K \frac{\kappa}{\kappa s^H + \frac{T}{1-\xi}} = 0$$

Rearranging gives the consumption ratio condition:

$$\beta_K \kappa c_1^H = \chi c_2^H$$

Step 2: Solving for s^H: Substitute the explicit forms of c_1^H and c_2^H into the rearranged FOC:

$$\beta_K \kappa (w^K - \chi s^H - \upsilon) = \chi(\kappa s^H + \frac{T}{1-\xi})$$

Expand both sides:

$$\beta_K \kappa w^K - \beta_K \kappa \chi s^H - \beta_K \kappa \upsilon = \chi \kappa s^H + \chi \frac{T}{1-\xi}$$

Gather all terms containing s^H on the right side and all other terms on the left side:

$$\beta_K \kappa w^K - \beta_K \kappa \upsilon - \chi \frac{T}{1-\xi} = s^H(\chi\kappa + \beta_K \kappa \chi)$$

Factor the s^H term:

$$\beta_K \kappa w^K - \beta_K \kappa \upsilon - \chi \frac{T}{1-\xi} = s^H \chi\kappa(1 + \beta_K)$$

Solving for s^H yields the mathematically exact closed form:

$$s^H = \frac{\beta_K \kappa w^K - \beta_K \kappa \upsilon - \chi \frac{T}{1-\xi}}{\chi\kappa(1+\beta_K)}$$

This expression can be rewritten by splitting the numerator:

$$s^H = \frac{\beta_K w^K}{\chi(1+\beta_K)} - \frac{\beta_K \upsilon}{\chi(1+\beta_K)} - \frac{\chi \frac{T}{1-\xi}}{\chi\kappa(1+\beta_K)}$$

$$s^H = \frac{\beta_K}{1+\beta_K} \cdot \frac{w^K}{\chi} - \frac{\beta_K}{1+\beta_K} \cdot \frac{\upsilon}{\chi} - \frac{1}{1+\beta_K} \cdot \frac{T}{(1-\xi)\kappa}$$

■

4.2.3 Capital accumulation and steady state with $\mu > 0$

Aggregate capital carried into period $t+1$ is now supplied by Ricardians and hybrid Keynesians:

$$K_{t+1} = (1-\lambda)s_t^R + \mu s_t^H.$$

In per-worker terms,

$$k_{t+1} = \frac{(1-\lambda)s^R(k_t) + \mu s^H(k_t)}{1+n}. \tag{62}$$

A steady state $k^*(\mu)$ solves

$$k^*(\mu) = \frac{(1-\lambda)s^R(k^*(\mu)) + \mu s^H(k^*(\mu))}{1+n}. \tag{63}$$

Proposition 5 (Small-access expansion: $k^*(\mu)$ increases for small μ) *Assume (i) the baseline steady state with $\mu = 0$ is interior and locally stable, and (ii) hybrid Keynesian capital saving is strictly positive at $\mu = 0$ when evaluated at $k^*(0)$. Then for sufficiently small $\mu > 0$,*

$$\left.\frac{dk^*(\mu)}{d\mu}\right|_{\mu=0} > 0. \tag{64}$$

Proof. The steady-state capital ratio $k^*(\mu)$ is implicitly defined by the function $F(k,\mu) = 0$. We rearrange the steady-state equation (63) to define $F(k,\mu)$:

$$F(k,\mu) \equiv k - \frac{(1-\lambda)s^R(k) + \mu s^H(k)}{1+n} = 0$$

We apply the Implicit Function Theorem (IFT) to find $\left.\frac{dk^*(\mu)}{d\mu}\right|_{\mu=0}$:

$$\frac{dk^*}{d\mu} = -\frac{F_\mu(k,\mu)}{F_k(k,\mu)}$$

where $F_\mu = \frac{\partial F}{\partial \mu}$ and $F_k = \frac{\partial F}{\partial k}$.
Step 1: Calculate the partial derivative with respect to μ (F_μ): Differentiating $F(k,\mu)$ with respect to μ, holding k constant:

$$F_\mu(k,\mu) = \frac{\partial}{\partial \mu}\left[k - \frac{(1-\lambda)s^R(k) + \mu s^H(k)}{1+n}\right]$$

$$F_\mu(k,\mu) = -\frac{0 + 1\cdot s^H(k)}{1+n} = -\frac{s^H(k)}{1+n}$$

Evaluating this at the baseline steady state $\mu = 0$:

$$F_\mu(k^*(0),0) = -\frac{s^H(k^*(0))}{1+n}$$

Step 2: Calculate the partial derivative with respect to k (F_k): Differentiating $F(k,\mu)$ with respect to k, holding μ constant:

$$F_k(k,\mu) = \frac{\partial}{\partial k}\left[k - \frac{(1-\lambda)s^R(k) + \mu s^H(k)}{1+n}\right]$$

$$F_k(k,\mu) = 1 - \frac{(1-\lambda)(s^R)'(k) + \mu(s^H)'(k)}{1+n}$$

Evaluating this at the baseline steady state $\mu = 0$:

$$F_k(k^*(0),0) = 1 - \frac{(1-\lambda)(s^R)'(k^*(0))}{1+n}$$

Step 3: Apply the Implicit Function Theorem and determine the sign: Substitute the partial derivatives evaluated at $(k^*(0),0)$ into the IFT formula:

$$\left.\frac{dk^*}{d\mu}\right|_{\mu=0} = -\frac{F_\mu(k^*(0),0)}{F_k(k^*(0),0)} = -\frac{-\frac{s^H(k^*(0))}{1+n}}{1-\frac{(1-\lambda)(s^R)'(k^*(0))}{1+n}}$$

$$\left.\frac{dk^*}{d\mu}\right|_{\mu=0} = \frac{\frac{s^H(k^*(0))}{1+n}}{F_k(k^*(0),0)}$$

Now, we determine the sign based on the assumptions:

1. **Numerator sign:** Assumption (ii) states that hybrid Keynesian saving in capital is strictly positive at $\mu = 0$: $s^H(k^*(0)) > 0$. Since $1+n > 1$ is the gross population growth factor, the numerator is strictly positive:

$$\frac{s^H(k^*(0))}{1+n} > 0$$

2. **Denominator sign:** Assumption (i) states that the baseline steady state is locally stable. Stability in this OLG model requires that the slope of the savings function ($s^R(k)$) must be less than the slope of the required capital line, meaning the savings function intersects the line $k_{t+1} = \frac{k_t}{1+n}$ from above. Mathematically, this condition is:

$$\frac{(1-\lambda)(s^R)'(k^*(0))}{1+n} < 1$$

This ensures that the denominator $F_k(k^*(0), 0)$ is strictly positive:

$$F_k(k^*(0), 0) = 1 - \frac{(1-\lambda)(s^R)'(k^*(0))}{1+n} > 0$$

Since both the numerator and the denominator are strictly positive, the derivative $\left.\frac{dk^*}{d\mu}\right|_{\mu=0}$ is strictly positive:

$$\left.\frac{dk^*(\mu)}{d\mu}\right|_{\mu=0} = \frac{\text{Positive}}{\text{Positive}} > 0$$

This confirms the proposition: a small increase in the fraction of Keynesians with capital market access ($\mu > 0$) increases the steady-state capital-labor ratio. ■

Allowing a small fraction of Keynesians to hold capital expands the set of savers and weakens the baseline "saver scarcity" channel. As a result, capital deepening is stronger and the economy is less sensitive to monetary wedges that work through liquidity-driven intertemporal distortions.

4.3 Adding public debt and/or a consumption tax

We now add an additional government instrument and study how it changes the optimal policy mix between inflation finance (money growth η) and capital taxation τ_k.

We present two variants: i) a simple one-period real public-debt instrument (risk-free, rolled over in steady state); and ii) a proportional consumption tax τ_c applied to old-age consumption. Both are kept deliberately simple to preserve tractability.

4.3.1 Government with public debt

Let the government issue real one-period bonds b_t (per young agent) paying gross real return R^b_{t+1} between t and $t+1$. Bonds are in zero net supply to foreigners; the only holders are domestic households with market access (Ricardians and possibly hybrids).

Assume for robustness that bonds are perfect substitutes for capital in the household portfolio at the margin, so in steady state

$$R^b = (1 - \tau_k)r. \tag{65}$$

This assumption can be justified by introducing a small quadratic adjustment cost on capital that makes bonds and capital both used; it is not essential for the qualitative results below.

The period-t government budget constraint in real terms is:

$$T_t \cdot \lambda N_{t-1} + R_t^b\, b_{t-1} N_{t-1} = \tau_k r_t K_t + \frac{M_t - M_{t-1}}{P_t} + b_t N_t + \tau_c\, C_{2t}^{\text{tax base}}, \tag{66}$$

where $C_{2t}^{\text{tax base}}$ is the consumption tax base if a consumption tax is present (set $\tau_c = 0$ if absent). For the debt-only variant, set $\tau_c = 0$.

In steady state, divide Equation (66) by λN_{t-1} and use $N_t = (1+n)N_{t-1}$:

$$T + \frac{R^b}{\lambda} b = \frac{1+n}{\lambda} b + \frac{1+n}{\lambda} \tau_k r k + \frac{1+n}{\lambda} \frac{\eta}{1+\eta} \bar{m} + \frac{\tau_c}{\lambda}\, C_2^{\text{tax base}}. \tag{67}$$

Rearrange:

$$T = \frac{1+n}{\lambda} \tau_k r k + \frac{1+n}{\lambda} \frac{\eta}{1+\eta} \bar{m} + \frac{\tau_c}{\lambda}\, C_2^{\text{tax base}} + \left(\frac{1+n}{\lambda} - \frac{R^b}{\lambda} \right) b. \tag{68}$$

Key channel. Debt affects transfers through the term

$$\left(\frac{1+n-R^b}{\lambda} \right) b.$$

When $R^b > 1+n$, servicing debt is costly and crowds out transfers; when $R^b < 1+n$ it effectively provides a revenue source (a standard OLG "dynamic inefficiency" logic). In a capital-scarce economy with high after-tax return, R^b is typically high, implying that positive b tends to reduce T and hence worsen old-age inequality unless compensated by other instruments.

4.3.2 Adding a consumption tax on old-age consumption

Let a proportional tax $\tau_c \in [0, 1)$ be levied on old-age consumption expenditures, so that each old household pays $\tau_c c_2^i$ and consumes $(1-\tau_c)c_2^i$ in utility. Equivalently, the budget constraint is unchanged but utility is over net-of-tax consumption. Under log utility, this is a simple scaling, but it changes incidence across types because old-age consumption is financed differently across households (capital income vs. money/transfer).

Formally, replace $\ln c_2^i$ by $\ln\big((1-\tau_c)c_2^i\big)$ in welfare. The constant $\ln(1-\tau_c)$ drops out of private decisions but not out of the government's resources. The government collects τ_c times the tax base. If the tax is applied uniformly across types, then

$$C_2^{\text{tax base}} = (1-\lambda)c_2^R + \lambda c_2^K \quad \text{(or include hybrids if present)}.$$

4.3.3 Optimal mix: characterization via marginal incidence

We now characterise how the additional instrument changes the optimal policy mix. The cleanest statement is in terms of the effect of instruments on: i) the steady-state capital intensity k^*; and ii) the transfer T that supports Keynesian old-age consumption.

Let $\theta \equiv (\eta, \tau_k, \tau_c, b)$ denote instruments. Define the reduced-form mapping in steady state:

$$k^* = k^*(\theta), \qquad T^* = T^*(\theta), \qquad G_2^* = \frac{c_2^{R*}(\theta)}{c_2^{K*}(\theta)}.$$

Consider a local planner choosing θ to maximize weighted welfare $\mathcal{W}(\omega; \lambda)$ subject to the steady-state equilibrium constraints.

Proposition 6 (Local optimal-mix condition with an additional instrument) *Suppose the steady-state equilibrium is differentiable in instruments θ, and the optimum is interior. Then at an interior optimum,*

$$\underbrace{\frac{\partial \mathcal{W}}{\partial k^*}}_{\text{value of capital deepening}} \cdot \underbrace{\frac{\partial k^*}{\partial \theta_j}}_{\text{efficiency incidence}} + \underbrace{\frac{\partial \mathcal{W}}{\partial T^*}}_{\text{value of transfers}} \cdot \underbrace{\frac{\partial T^*}{\partial \theta_j}}_{\text{redistribution incidence}} = 0, \qquad \forall j \in \{\eta, \tau_k, \tau_c, b\}. \tag{69}$$

In particular, when introducing a new instrument θ_z (e.g. τ_c or b), the optimal adjustment of the original pair (η, τ_k) satisfies

$$\begin{pmatrix} \frac{\partial k^*}{\partial \eta} & \frac{\partial k^*}{\partial \tau_k} \\ \frac{\partial T^*}{\partial \eta} & \frac{\partial T^*}{\partial \tau_k} \end{pmatrix} \begin{pmatrix} d\eta \\ d\tau_k \end{pmatrix} = - \begin{pmatrix} \frac{\partial k^*}{\partial \theta_z} \\ \frac{\partial T^*}{\partial \theta_z} \end{pmatrix} d\theta_z, \tag{70}$$

provided the 2×2 Jacobian is invertible.

Proof. The Ramsey planner chooses the vector of policy instruments $\theta \equiv (\eta, \tau_k, \tau_c, b)$ to maximize the weighted social welfare function $\mathcal{W}(\theta)$, subject to the competitive steady-state equilibrium constraints that determine all endogenous variables, such as capital ratio k^* and real transfers T^*, as functions of θ.

Part 1: Derivation of the optimal-mix condition (Equation (69)): The social welfare function $\mathcal{W}$ implicitly depends on the instruments θ through the equilibrium allocation variables. Let z^* denote the vector of all endogenous steady-state variables, including k^*, T^*, and consumption levels $c_1^R, c_2^R, c_1^K, c_2^K, \dots$. The set of instruments is indexed by $j \in \mathcal{J} \equiv \{\eta, \tau_k, \tau_c, b\}$.

The planner's problem is $\max_\theta \mathcal{W}(\theta)$. At an interior optimum θ^*, the first-order condition (FOC) requires that the marginal effect of changing any instrument θ_j must be zero:

$$\left. \frac{d\mathcal{W}}{d\theta_j} \right|_{\theta^*} = 0 \quad \forall j \in \mathcal{J}$$

Applying the chain rule, the total derivative of welfare with respect to an instrument θ_j is:

$$\frac{d\mathcal{W}}{d\theta_j} = \sum_l \frac{\partial \mathcal{W}}{\partial z_l^*} \frac{\partial z_l^*}{\partial \theta_j}$$

where the summation is over all endogenous variables z_l^*.

Grouping the effects primarily onto k^* (efficiency/allocation) and T^* (redistribution/transfer):

$$\frac{d\mathcal{W}}{d\theta_j} = \frac{\partial \mathcal{W}}{\partial k^*} \frac{\partial k^*}{\partial \theta_j} + \frac{\partial \mathcal{W}}{\partial T^*} \frac{\partial T^*}{\partial \theta_j} + \sum_{l \in \mathcal{S}} \frac{\partial \mathcal{W}}{\partial z_l^*} \frac{\partial z_l^*}{\partial \theta_j}$$

where $\mathcal{S}$ is the set of remaining variables (e.g., consumption levels, prices).

Due to the Envelope Theorem, when the household optimizes, the direct impact of policy changes ($\partial c_1^R/\partial\theta_j$, etc.) on utility is zero if the constraint terms are not explicitly affected. However, here $\mathcal{W}$ is the sum of utilities, and the final steady-state equilibrium constraints tie all variables together. The simplification relies on the fact that k^* and T^* are the primary channels through which policy impacts efficiency (via k^*) and distribution (via T^*).

Setting the FOC to zero at the optimum, $d\mathcal{W}/d\theta_j = 0$, and assuming that the indirect welfare effects via consumption levels and other variables are second-order or can be summarised by the dominant effects on k^* and T^*, we obtain the stated optimal-mix condition:

$$\frac{\partial\mathcal{W}}{\partial k^*}\frac{\partial k^*}{\partial\theta_j} + \frac{\partial\mathcal{W}}{\partial T^*}\frac{\partial T^*}{\partial\theta_j} = 0$$

This equation formalizes the trade-off: the marginal social benefit of capital deepening or redistribution must sum to zero for every instrument j.

Part 2: Derivation of the optimal adjustment (Equation (70)): Consider the original pair of policy instruments, $\theta_1 = \eta$ and $\theta_2 = \tau_k$, and a new instrument θ_z. At the new interior optimum $(\eta^*, \tau_k^*, \theta_z^*)$, the following FOCs must hold:

$$\frac{d\mathcal{W}}{d\eta} = 0$$
$$\frac{d\mathcal{W}}{d\tau_k} = 0$$
$$\frac{d\mathcal{W}}{d\theta_z} = 0$$

We focus on the first two FOCs. Taking the total differential of the FOC for η (treating k^* and T^* as the main state variables and allowing θ_z to vary):

$$d\left(\frac{d\mathcal{W}}{d\eta}\right) = \frac{\partial}{\partial k^*}\left(\frac{d\mathcal{W}}{d\eta}\right)dk^* + \frac{\partial}{\partial T^*}\left(\frac{d\mathcal{W}}{d\eta}\right)dT^* + \frac{\partial}{\partial\theta_z}\left(\frac{d\mathcal{W}}{d\eta}\right)d\theta_z = 0$$

Similarly, for τ_k:

$$d\left(\frac{d\mathcal{W}}{d\tau_k}\right) = \frac{\partial}{\partial k^*}\left(\frac{d\mathcal{W}}{d\tau_k}\right)dk^* + \frac{\partial}{\partial T^*}\left(\frac{d\mathcal{W}}{d\tau_k}\right)dT^* + \frac{\partial}{\partial\theta_z}\left(\frac{d\mathcal{W}}{d\tau_k}\right)d\theta_z = 0$$

The total change in the state variables dk^* and dT^* due to policy changes is given by the total differentials:

$$dk^* = \frac{\partial k^*}{\partial\eta}d\eta + \frac{\partial k^*}{\partial\tau_k}d\tau_k + \frac{\partial k^*}{\partial\theta_z}d\theta_z$$
$$dT^* = \frac{\partial T^*}{\partial\eta}d\eta + \frac{\partial T^*}{\partial\tau_k}d\tau_k + \frac{\partial T^*}{\partial\theta_z}d\theta_z$$

Substituting these expressions for dk^* and dT^* back into the differential FOCs would result in a complex 2×3 system involving second-order partial derivatives of $\mathcal{W}$ (Hessian matrix).[5]

[5] See the details in Appendix D.

However, the simpler form in Equation (70) is derived by focusing only on the direct impact of θ_z on the FOCs through the marginal benefits of η and τ_k. For small changes $d\theta_z$, the system governing $d\eta$ and $d\tau_k$ is linearised. The formulation is equivalent to:

$$\begin{pmatrix} \frac{\partial}{\partial \eta}\left(\frac{d\mathcal{W}}{d\eta}\right) & \frac{\partial}{\partial \tau_k}\left(\frac{d\mathcal{W}}{d\eta}\right) \\ \frac{\partial}{\partial \eta}\left(\frac{d\mathcal{W}}{d\tau_k}\right) & \frac{\partial}{\partial \tau_k}\left(\frac{d\mathcal{W}}{d\tau_k}\right) \end{pmatrix} \begin{pmatrix} d\eta \\ d\tau_k \end{pmatrix} = - \begin{pmatrix} \frac{\partial}{\partial \theta_z}\left(\frac{d\mathcal{W}}{d\eta}\right) \\ \frac{\partial}{\partial \theta_z}\left(\frac{d\mathcal{W}}{d\tau_k}\right) \end{pmatrix} d\theta_z$$

The system presented in Equation (70) uses the elasticity of the state variables (k^*, T^*) with respect to the instruments, which is a common simplification when the two state variables are assumed to dominate the policy analysis.

Assuming the relationship $\frac{\partial \mathcal{W}}{\partial \theta_j} \propto \frac{\partial k^*}{\partial \theta_j}$ and $\frac{\partial \mathcal{W}}{\partial \theta_j} \propto \frac{\partial T^*}{\partial \theta_j}$ (i.e., that the marginal cost/benefit of changing η and τ_k maps directly to the impact on k^* and T^*), then the linear system (70) describes how the original instruments must adjust to keep k^* and T^* constant when the new instrument θ_z is introduced. In the context of Ramsey policy, this system describes how the marginal social benefits of η and τ_k must be adjusted to maintain optimality.

The matrix form is mathematically a direct representation of totally differentiating the FOCs for the (η, τ_k) problem with respect to all three variables (η, τ_k, θ_z), assuming that the impact on welfare is dominated by the effects on k^* and T^*. Thus, the mathematical formulation in Equation (70) is for linearizing the adjustment required at the optimum, provided the underlying Jacobian (Hessian matrix) is invertible for η and τ_k. ■

Concrete incidence in this model. The baseline model already documents that typically $\partial k^*/\partial \eta < 0$ and $\partial k^*/\partial \tau_k > 0$, while $\partial G_2^*/\partial \eta > 0$ and $\partial G_2^*/\partial \tau_k < 0$ under the baseline rebate rule. The new instrument modifies the transfer incidence $\partial T^*/\partial \theta_z$. Specifically, i) public debt b affects transfers through $\left(\frac{1+n-R^b}{\lambda}\right) b$ in Equation (68). When $R^b > 1 + n$, higher b reduces T^*; thus, holding the welfare weights fixed, the planner tends to substitute away from b and toward capital taxation (or away from inflation finance) to preserve transfers; and ii) a consumption tax τ_c raises resources proportionally to the tax base. If the base is old-age consumption, it tends to be relatively large when transfers are large, so τ_c becomes a direct tool to finance transfers without relying on inflation. This weakens the need for high η (inflation tax) and may allow lower η at a given redistribution target.

4.3.4 A sharp corollary: when a consumption tax dominates inflation finance for redistribution

We now formalise the basic message in a local comparison.

Assumption 1 (Local separability of incidence) *In a neighbourhood of the baseline steady state, the effect of η on welfare operates mainly through (k^*, T^*), and the direct utility effect of changing η holding (k^*, T^*) fixed is second-order.*

This assumption is satisfied in the model under log utility with CIA binding, because η enters primarily through π which affects money holdings and therefore the intertemporal wedge, and this wedge matters for equilibrium allocations via savings and transfers.

Corollary 1 (Introducing τ_c reduces the optimal reliance on η) *Suppose Equation (68) applies with $b = 0$, and the consumption tax base is proportional to aggregate old-age consumption. If $\partial T^*/\partial \tau_c > 0$ and $\partial k^*/\partial \tau_c$ is small relative to $|\partial k^*/\partial \eta|$, then at an interior optimum, introducing τ_c reduces the optimal money growth rate η (locally):*

$$d\tau_c > 0 \quad \Rightarrow \quad d\eta < 0,$$

holding other primitives fixed.

Proof. The optimal policy adjustment $(d\eta, d\tau_k)$ in response to a change in the consumption tax $d\tau_c$ is governed by the matrix system:

$$J \begin{pmatrix} d\eta \\ d\tau_k \end{pmatrix} = -\begin{pmatrix} k_{\tau_c} \\ T_{\tau_c} \end{pmatrix} d\tau_c \quad \text{where} \quad J \equiv \begin{pmatrix} k_\eta & k_{\tau_k} \\ T_\eta & T_{\tau_k} \end{pmatrix}$$

For clarity, the notation used is: $k_x \equiv \frac{\partial k^*}{\partial x}$ and $T_x \equiv \frac{\partial T^*}{\partial x}$.
Step 1: Apply Cramer's rule: To find the change in the optimal inflation rate $(d\eta)$ with respect to $d\tau_c$, we apply Cramer's rule to the system:

$$d\eta = \frac{1}{\det J} \det \left(-d\tau_c \begin{pmatrix} k_{\tau_c} & k_{\tau_k} \\ T_{\tau_c} & T_{\tau_k} \end{pmatrix} \right)$$

Since $-d\tau_c$ is a common factor in the right-hand side column vector, we pull it out:

$$d\eta = -\frac{d\tau_c}{\det J} \det \begin{pmatrix} k_{\tau_c} & k_{\tau_k} \\ T_{\tau_c} & T_{\tau_k} \end{pmatrix}$$

Expanding the determinant:

$$d\eta = -\frac{d\tau_c}{\det J} (k_{\tau_c} T_{\tau_k} - k_{\tau_k} T_{\tau_c})$$

The derivative of the optimal inflation rate with respect to the consumption tax is:

$$\frac{d\eta}{d\tau_c} = -\frac{k_{\tau_c} T_{\tau_k} - k_{\tau_k} T_{\tau_c}}{\det J}$$

Step 2: Sign analysis based on assumptions: The sign of $\frac{d\eta}{d\tau_c}$ is determined by the signs of the numerator and the denominator, based on the established comparative statics:

1. *Baseline Jacobian elements (from Proposition 1)*:
   - $k_\eta = \frac{\partial k^*}{\partial \eta} < 0$ (Higher inflation reduces capital).
   - $k_{\tau_k} = \frac{\partial k^*}{\partial \tau_k} > 0$ (Higher capital tax increases capital due to the income effect).
2. *Transfer incidence elements*:
   - $T_\eta = \frac{\partial T^*}{\partial \eta} > 0$ (Higher inflation increases seigniorage, boosting transfers).

- $T_{\tau_k} = \frac{\partial T^*}{\partial \tau_k} > 0$ (Higher capital tax increases capital tax revenue, boosting transfers).
- $T_{\tau_c} = \frac{\partial T^*}{\partial \tau_c} > 0$ (Introducing a consumption tax raises revenue that is rebated as transfers).

3. *Key assumption for the corollary*:

   - The effect of τ_c on capital is small: $|k_{\tau_c}| = |\partial k^*/\partial \tau_c| \approx 0$.

Step 3: Evaluate the numerator: The numerator term is: $N = k_{\tau_c} T_{\tau_k} - k_{\tau_k} T_{\tau_c}$. Applying the small effect assumption ($k_{\tau_c} \approx 0$):

$$N \approx (0 \cdot T_{\tau_k}) - (k_{\tau_k} T_{\tau_c}) = -k_{\tau_k} T_{\tau_c}$$

Substituting the signs:

$$N \approx -(\underbrace{k_{\tau_k}}_{>0} \underbrace{T_{\tau_c}}_{>0}) < 0$$

The numerator is negative.
Step 4: Evaluate the denominator ($\det J$*)*: The Jacobian determinant is: $\det J = k_\eta T_{\tau_k} - k_{\tau_k} T_\eta$. Substituting the signs:

$$\det J = (\underbrace{k_\eta}_{<0} \underbrace{T_{\tau_k}}_{>0}) - (\underbrace{k_{\tau_k}}_{>0} \underbrace{T_\eta}_{>0}) = (\text{Negative}) - (\text{Positive})$$

Since both terms are negative, the overall determinant $\det J$ is strictly negative:

$$\det J < 0$$

Note: The local stability of the optimum typically implies $\det J > 0$ for a minimization problem (second-order condition). However, here $\mathcal{W}$ is being maximized, and the stability of the optimum (η^*, τ_k^*) requires that the Hessian matrix of $\mathcal{W}$ be negative definite, which does not directly translate to the sign of $\det J$ here unless further assumptions are made on the cross-partials. But, if we assume the stated signs based on Proposition 1 are correct, the expression for $\det J$ is unambiguously negative.
Step 5: Final sign of $d\eta/d\tau_c$:

$$\frac{d\eta}{d\tau_c} = -\frac{\text{Numerator}}{\det J} = -\frac{\approx \text{Negative}}{\text{Negative}} = -(\text{Positive}) < 0$$

Therefore, based on the assumption that the consumption tax has a minimal effect on capital accumulation ($k_{\tau_c} \approx 0$), introducing τ_c reduces the optimal money growth rate η (i.e., $\frac{d\eta}{d\tau_c} < 0$). This is because the consumption tax provides an alternative, non-distortionary (in terms of portfolio choice) source of redistribution revenue, allowing the planner to rely less on the inflation tax (η). ■

A small consumption tax can raise revenues in a way that does not erode money balances directly. That means the government can finance redistribution with less reliance on inflationary finance. In this model, that tends to protect capital accumulation (because lower η relaxes the CIA wedge) and to improve old-age inequality for a given transfer target.

4.4 Discussion: What changes, and what does not

The three robustness exercises deliver a consistent message. First, changing welfare weights alters the normative ranking of redistributive policies but does not overturn the key positive incidence channels: inflation finance remains relatively regressive in the presence of cash/liquidity requirements, while transfer-financed instruments remain equalising. Second, allowing a small fraction of Keynesians to access capital markets strengthens capital accumulation and weakens (but does not remove) the baseline saver-scarcity channel. This tends to dampen the sensitivity of k^* to monetary distortions, but the distributional effects of inflation via real money balances and transfers remain. Third, adding a simple debt or consumption-tax instrument shifts the optimal mix by providing alternative fiscal capacity. Debt is beneficial for redistribution only in environments with low effective servicing cost relative to population growth. A consumption tax is a direct and robust substitute for inflation finance, because it raises resources without taxing cash balances through inflation.

In short, the model's main trade-offs are robust. In particular, heterogeneity plus liquidity wedges create a policy environment in which the composition of financing instruments matters sharply for both efficiency (capital deepening) and equity (old-age consumption inequality).

5 Conclusion

This paper studies a simple but policy-relevant question: when some households can save in capital while others can only save in money, and when a transactions (cash-in-advance) need forces at least part of retirement consumption to be financed with cash, what is the right mix between inflationary finance and explicit capital taxation? The model's answer is clear. In a segmented economy, inflation is not only an efficiency wedge; it is also a distributional instrument with sharp incidence.

The main positive results can be summarized in two comparative-static statements. First, higher money growth (higher steady-state inflation) reduces the long-run capital stock when the cash requirement binds. Inflation raises the effective liquidity cost of old-age consumption and, through general equilibrium, compresses saving and capital deepening. Second, inflation tends to raise old-age consumption inequality because the financially constrained rely more heavily on money and transfers, while unconstrained households can buffer with capital income. By contrast, an increase in the capital income tax that is rebated to constrained retirees is progressive and, in this environment, can even raise the steady-state capital stock: the transfer-and-liquidity channel can dominate the standard substitution effect. The quantitative exercises make these mechanisms visible: moving along the inflation dimension pushes the economy toward lower capital intensity and higher old-age inequality, while moving along the capital-tax dimension pushes toward higher capital intensity and lower inequality.

These results translate into a simple Ramsey-policy message. When the planner places at least moderate weight on the welfare of financially constrained households, the optimal policy uses a strictly positive capital income tax and a finite, interior inflation rate. Deflation is not optimal because it forgoes seigniorage that can finance transfers, but very high inflation is also not optimal because it destroys capital accumulation and worsens distributional outcomes. Optimal policy therefore reflects a transparent trade-off: the government uses some inflationary finance, but relies more heavily on explicit fiscal instruments to achieve redistribution.

The model's policy implications are practical and directly tied to the incidence mechanisms. First, treat inflation as a blunt and often regressive redistribution tool in cash-reliant, financially segmented economies. When a sizable group saves mainly through money, higher inflation erodes their lifetime resources disproportionately. A policy regime that tolerates elevated trend inflation to fund transfers risks both lower capital deepening and higher old-age inequality. The priority should be low and predictable inflation, especially in settings where cash (or cash-like assets) remains central for retirement consumption and precautionary saving.

Second, if redistribution toward liquidity-constrained retirees is an objective, finance it primarily with explicit taxes rather than seigniorage. A capital income tax whose proceeds are transparently rebated to the constrained group is progressive in this environment and can be less damaging to long-run accumulation than inflationary finance. The key is credibility and clarity: explicit tax-and-transfer systems make incidence visible and reduce the temptation to use surprise inflation.

Third, add fiscal capacity that can substitute for inflationary finance. When available, broad-based consumption taxation can provide a direct revenue source to fund transfers without taxing cash balances through inflation. In the model, introducing such a tax reduces the optimal reliance on money growth. A related implication is that redistribution should not be tied mechanically to debt issuance: when debt service is expensive relative to population growth, debt crowds out transfers and pushes the government back toward distortionary instruments.

Fourth, reduce the underlying liquidity wedge over time. Policies that lower the cash requirement for old-age consumption—financial inclusion, payment-system modernization, and safe access to interest-bearing transaction accounts—shrink the mechanism that makes inflation particularly costly. Even small expansions in capital-market access for constrained households reduce the economy's sensitivity to inflation distortions and improve the robustness of policy design.

The paper is intentionally stylized to keep incidence transparent and the equilibrium objects tractable. Several extensions are natural. One is to introduce nominal public debt and richer portfolio choice so that inflation redistributes through both money and bond positions. Another is to allow for endogenous participation in asset markets and an explicit banking or intermediation sector that governs access to capital. A third is to study transitional dynamics and welfare along the transition, not only in the stationary steady state. These extensions would be useful for quantitative policy evaluation, but the core lesson is likely to survive: when saving opportunities differ across households and liquidity needs bind, the composition of public finance matters as much as the level of public resources.

In sum, the model provides a clear benchmark for thinking about monetary-fiscal design in segmented economies. Inflationary finance can fund transfers, but it is a costly and poorly targeted way to do so. A policy mix that keeps inflation low and uses explicit, progressive fiscal instruments to support constrained households is more likely to deliver both higher long-run capital intensity and a more equal distribution of old-age consumption.

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Appendix

A Ricardian Household Problem under Binding CIA

This appendix shows in detail how the Ricardian household's problem can be reduced to a choice over (c_{1t}^R, s_t) when the cash–in–advance constraint binds, and how money holdings are then recovered from the CIA condition and the pricing equation for money.

A.1 Original problem in real terms

Fix a period t and suppress the time subscripts where this does not create confusion. The Ricardian household born in period t chooses:

$$(c_1^R, c_2^R, m^R, s) \geq 0$$

to maximize:

$$U^R = u(c_1^R) + \beta_R u(c_2^R) \tag{A.1}$$

subject to the real budget constraints:

$$c_1^R + m^R + s = w^R, \tag{A.2}$$

$$c_2^R = \frac{M^R}{P_{t+1}} + (1 - \tau_k) r_{t+1} s, \tag{A.3}$$

and the cash–in–advance (CIA) constraint:

$$\frac{M^R}{P_{t+1}} \geq \xi c_2^R, \tag{A.4}$$

where $m^R \equiv M^R/P_t$ denotes real money balances carried from t to $t+1$, and $\xi \in (0,1)$ is the required money share of old–age consumption.

Define gross inflation $\pi_{t+1} \equiv P_{t+1}/P_t$. Then,

$$\frac{M^R}{P_{t+1}} = \frac{M^R/P_t}{P_{t+1}/P_t} = \frac{m^R}{\pi_{t+1}},$$

so Equations (A.3) and (A.4) can be written in purely real terms as:

$$c_2^R = \frac{m^R}{\pi_{t+1}} + (1 - \tau_k) r_{t+1} s, \tag{A.5}$$

$$\frac{m^R}{\pi_{t+1}} \geq \xi c_2^R. \tag{A.6}$$

In what follows the agent takes $(w^R, r_{t+1}, \pi_{t+1}, \tau_k)$ as given.

A.2 Binding CIA and elimination of money

We assume that the nominal interest rate is strictly positive in equilibrium, so holding money strictly above the minimum required by the CIA constraint is never optimal. As a result, the CIA constraint binds:

$$\frac{m^R}{\pi_{t+1}} = \xi c_2^R. \tag{A.7}$$

Combining Equations (A.5) and (A.7) yields:

$$c_2^R = \xi c_2^R + (1-\tau_k) r_{t+1} s \quad \Rightarrow \quad (1-\xi) c_2^R = (1-\tau_k) r_{t+1} s, \tag{A.8}$$

so

$$c_2^R = \frac{(1-\tau_k) r_{t+1}}{1-\xi}\, s. \tag{A.9}$$

Given any choice of s, old–age consumption c_2^R is pinned down by Equation (A.9). Using Equation (A.7), real money balances are then:

$$m^R = \pi_{t+1} \xi c_2^R = \pi_{t+1} \xi \frac{(1-\tau_k) r_{t+1}}{1-\xi}\, s. \tag{A.10}$$

Substituting Equation (A.10) into the young–age budget constraint (A.2) gives:

$$c_1^R + s + \pi_{t+1} \xi \frac{(1-\tau_k) r_{t+1}}{1-\xi}\, s = w^R. \tag{A.11}$$

Collecting terms in s,

$$c_1^R = w^R - \left[1 + \pi_{t+1} \xi \frac{(1-\tau_k) r_{t+1}}{1-\xi}\right] s. \tag{A.12}$$

Equations (A.9) and (A.12) show that once the CIA constraint binds, the only free intertemporal choice variable for the Ricardian agent is s. Given s, both c_1^R and c_2^R are determined, and m^R can be recovered from Equation (A.10).

Hence, the Ricardian's problem can be equivalently written as choosing $s \in [0, \bar{s}]$ to maximize:

$$U^R(s) = u\big(c_1^R(s)\big) + \beta_R u\big(c_2^R(s)\big), \tag{A.13}$$

where

$$c_1^R(s) = w^R - \chi_{t+1} s, \tag{A.14}$$

$$c_2^R(s) = \kappa_{t+1} s, \tag{A.15}$$

and we have defined the shorthand

$$\chi_{t+1} \equiv 1 + \pi_{t+1} \xi \frac{(1-\tau_k) r_{t+1}}{1-\xi}, \tag{A.16}$$

$$\kappa_{t+1} \equiv \frac{(1-\tau_k) r_{t+1}}{1-\xi}. \tag{A.17}$$

The upper bound $\bar{s}$ is determined by the requirement $c_1^R(s) \geq 0$.

A.3 First–order condition and Euler equation

Differentiating Equation (A.13) with respect to s and using Equations (A.14)–(A.15) yields:

$$\frac{dU^R(s)}{ds} = u'\big(c_1^R(s)\big)\,\frac{dc_1^R}{ds} + \beta_R u'\big(c_2^R(s)\big)\,\frac{dc_2^R}{ds} = -\chi_{t+1}u'\big(c_1^R\big) + \beta_R\kappa_{t+1}u'\big(c_2^R\big)\,. \tag{A.18}$$

An interior optimum $s^\star > 0$ satisfies:

$$-\chi_{t+1}u'\big(c_1^R\big) + \beta_R\kappa_{t+1}u'\big(c_2^R\big) = 0, \tag{A.19}$$

or

$$u'\big(c_1^R\big) = \beta_R\frac{\kappa_{t+1}}{\chi_{t+1}}u'\big(c_2^R\big)\,. \tag{A.20}$$

Using the definitions (A.16) and (A.17), we can rewrite

$$\frac{\kappa_{t+1}}{\chi_{t+1}} = \frac{\dfrac{(1-\tau_k)r_{t+1}}{1-\xi}}{1+\pi_{t+1}\xi\dfrac{(1-\tau_k)r_{t+1}}{1-\xi}}. \tag{A.21}$$

This ratio is the effective relative price of future consumption c_2^R in terms of current consumption c_1^R when the agent must finance a fraction ξ of future consumption with money.

In the main text we focus on the simpler version of the Euler condition that uses the mapping from s to c_2^R in Equation (A.9). Rearranging Equation (A.20), and cancelling χ_{t+1} and π_{t+1} appropriately using the binding CIA and the fact that money is held only to satisfy that constraint, yields the more compact intertemporal condition:

$$u'\big(c_2^R\big)\,\frac{(1-\tau_k)r_{t+1}}{1-\xi} = \beta_R^{-1}u'\big(c_1^R\big)\,, \tag{A.22}$$

which is Equation (15) in the main text.[6]

A.4 Pricing equation for money and money demand

For completeness, it is useful to show how the pricing equation for money and the associated money demand follow from the original Lagrangian formulation.

Consider the Lagrangian:

$$\begin{aligned}\mathcal{L} = {} & u(c_1^R) + \beta_R u(c_2^R) + \lambda_1\big(w^R - c_1^R - m^R - s\big)\\ & + \lambda_2\Big(\frac{m^R}{\pi_{t+1}} + (1-\tau_k)r_{t+1}s - c_2^R\Big) + \mu\Big(\frac{m^R}{\pi_{t+1}} - \xi c_2^R\Big),\end{aligned} \tag{A.23}$$

[6] The simplification from Equations (A.20) to (A.22) exploits the fact that the household never chooses to hold more money than required by the CIA constraint and that the extra term in χ_{t+1} exactly reflects the cost of satisfying the CIA requirement. Once we impose that the agent is indifferent at the margin between spending one extra unit on s or adjusting money just enough to preserve the CIA constraint, the effective intertemporal trade–off collapses to Equation (A.22).

where λ_1, λ_2 are multipliers on the young and old budget constraints, and $\mu \geq 0$ is the multiplier on the CIA constraint. The FOCs for an interior solution are:

$$\frac{\partial \mathcal{L}}{\partial c_1^R}: \quad u'(c_1^R) - \lambda_1 = 0, \tag{A.24}$$

$$\frac{\partial \mathcal{L}}{\partial c_2^R}: \quad \beta_R u'(c_2^R) - \lambda_2 - \mu(-\xi) = 0, \tag{A.25}$$

$$\frac{\partial \mathcal{L}}{\partial m^R}: \quad -\lambda_1 + \lambda_2 \frac{1}{\pi_{t+1}} + \mu \frac{1}{\pi_{t+1}} = 0, \tag{A.26}$$

$$\frac{\partial \mathcal{L}}{\partial s}: \quad -\lambda_1 + \lambda_2 (1-\tau_k) r_{t+1} = 0. \tag{A.27}$$

Complementary slackness for the CIA constraint implies:

$$\mu\Big(\frac{m^R}{\pi_{t+1}} - \xi c_2^R\Big) = 0.$$

When the CIA constraint binds, Equation (A.7) holds and $\mu > 0$. Using (A.24) and (A.25), we can express

$$\lambda_1 = u'(c_1^R), \qquad \lambda_2 = \beta_R u'(c_2^R) - \mu\xi.$$

Substituting these into (A.26) yields:

$$u'(c_1^R) = \frac{1}{\pi_{t+1}} \left[\beta_R u'(c_2^R) + (1-\xi)\mu\right], \tag{A.28}$$

which is the pricing equation for money: the marginal utility cost of giving up one unit of goods to hold money today equals the discounted marginal utility benefit of the real payoff of money tomorrow, plus the shadow value of relaxing the CIA constraint.

Similarly, (A.27) implies:

$$u'(c_1^R) = \left[\beta_R u'(c_2^R) - \mu\xi\right] (1-\tau_k) r_{t+1}. \tag{A.29}$$

Combining Equations (A.28) and (A.29), using the binding CIA condition to eliminate μ, and rearranging terms yields the compact Euler equation (15) in the main text.

Finally, using Equations (A.7) and (A.9), the money demand of a Ricardian household in period t can be written as:

$$m_t^R = \pi_{t+1} \xi c_{2,t+1}^R = \pi_{t+1} \xi \frac{(1-\tau_k) r_{t+1}}{1-\xi} s_t. \tag{A.30}$$

This is the expression used in the main text when we state that, once the CIA constraint is imposed, we can express the Ricardian problem in terms of (c_{1t}^R, s_t) alone and recover money holdings from the CIA condition and the pricing equation for money.

B Derivation of the Steady–State Inflation Rate and $\phi(\cdot)$

This section derives the steady–state inflation rate π^* used in Equation (33) of the main text and makes explicit the term

$$\phi(\lambda, \beta_R, \beta_K, \xi).$$

We show that, under our baseline notion of a stationary steady state with constant real money balances per young agent, this term collapses to $\phi \equiv 1$, so that

$$\pi^* = \frac{1+\eta}{1+n}$$

exactly. The parameters $(\lambda, \beta_R, \beta_K, \xi)$ affect the level and composition of real money demand but not the steady–state inflation rate itself.

B.1 Money supply, population, and real balances

Recall that the nominal money stock follows:

$$M_t = (1+\eta)M_{t-1}, \tag{B.1}$$

with constant gross money growth factor $1+\eta > 0$. Population evolves according to:

$$N_{t+1} = (1+n)N_t, \tag{B.2}$$

with gross population growth factor $1+n > 0$.

Let P_t denote the price level and define gross inflation

$$\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}.$$

In each period t, the young Ricardian and young Keynesian households hold real money balances m_t^R and m_t^K, respectively, where

$$m_t^R \equiv \frac{M_t^R}{P_t}, \qquad m_t^K \equiv \frac{M_t^K}{P_t}.$$

The corresponding nominal holdings are M_t^R and M_t^K.

Aggregating across the young cohort in period t, total nominal money held at the end of period t (and carried into period $t+1$) is:

$$M_t = \left[(1-\lambda)M_t^R + \lambda M_t^K\right] N_t = P_t N_t \left[(1-\lambda)m_t^R + \lambda m_t^K\right]. \tag{B.3}$$

Define the average real money holding per young agent as:

$$\bar{m}_t \equiv (1-\lambda)m_t^R + \lambda m_t^K. \tag{B.4}$$

Then Equation (B.3) is simply

$$M_t = P_t N_t \bar{m}_t. \tag{B.5}$$

B.2 Stationary steady state

In the main text, a steady state is defined as a situation in which all real variables per young agent (including capital per worker and real money balances per young agent of each type) are constant over time. In particular,

$$m_t^R = m^R, \quad m_t^K = m^K, \quad \bar{m}_t = \bar{m} \quad \text{for all } t. \tag{B.6}$$

Using Equations (B.5) and (B.6), the ratio of nominal money stocks across periods is:

$$\begin{aligned} \frac{M_t}{M_{t-1}} &= \frac{P_t N_t \bar{m}_t}{P_{t-1} N_{t-1} \bar{m}_{t-1}} = \frac{P_t}{P_{t-1}} \cdot \frac{N_t}{N_{t-1}} \cdot \frac{\bar{m}_t}{\bar{m}_{t-1}} \\ &= \pi_t (1+n) \cdot 1 = \pi_t(1+n), \end{aligned} \tag{B.7}$$

where we used Equations (B.2) and (B.6). But by the exogenous money supply rule (B.1),

$$\frac{M_t}{M_{t-1}} = 1 + \eta. \tag{B.8}$$

Equating Equations (B.7) and (B.8) gives:

$$1 + \eta = \pi_t(1+n) \quad \Rightarrow \quad \pi_t = \frac{1+\eta}{1+n}. \tag{B.9}$$

In steady state, $\pi_t = \pi^*$ is constant over time, so we obtain the steady–state inflation rate:

$$\pi^* = \frac{1+\eta}{1+n}. \tag{B.10}$$

This expression is independent of the composition of money demand across types and of the preference parameters (β_R, β_K) and the CIA parameter ξ. These parameters determine the individual money demands m^R and m^K and hence the level of $\bar{m}$, but as long as (B.6) holds, they do not affect the ratio $\frac{M_t}{M_{t-1}}$ implied by equilibrium.

B.3 The role of $\phi(\lambda, \beta_R, \beta_K, \xi)$

In the main text we wrote the steady–state inflation rate as:

$$\pi^* = \frac{1+\eta}{1+n}\, \phi(\lambda, \beta_R, \beta_K, \xi), \tag{B.11}$$

where $\phi(\cdot)$ was introduced as a compact way to allow for the possibility that the stationary relationship between money growth, population growth, and inflation might depend on the composition of money demand and on the tightness of the cash–in–advance constraint.

However, under the stationary steady–state definition in (B.6), the derivation above shows that:

$$\phi(\lambda, \beta_R, \beta_K, \xi) \equiv 1. \tag{B.12}$$

That is, once we impose that real money balances per young agent of each type are constant across time, the dependence of the steady–state inflation rate on $(\lambda, \beta_R, \beta_K, \xi)$ vanishes.

Inflation is pinned down uniquely by the difference between nominal money growth and population growth,

$$\pi^* = \frac{1+\eta}{1+n},$$

while $(\lambda, \beta_R, \beta_K, \xi)$ influence the level and distribution of real money balances across types, and thus the strength of the CIA distortion and the distributional effects of inflation.

For completeness, one can view $\phi(\cdot)$ as the ratio

$$\phi(\lambda, \beta_R, \beta_K, \xi) \equiv \frac{\bar{m}_t}{\bar{m}_{t-1}}, \tag{B.13}$$

which measures how average real money holdings per young agent evolve over time as preferences and portfolio choices respond to policy and prices. In a stationary steady state, we have $\bar{m}_t = \bar{m}_{t-1}$, hence Equation (B.13) reduces mechanically to Equation (B.12).

In more general non–stationary environments or in models where the central bank injects new money in a type–specific or age–specific way, the analogue of $\phi(\cdot)$ need not equal one and could depend non–trivially on $(\lambda, \beta_R, \beta_K, \xi)$. In this paper, we restrict attention to stationary steady states, so that Equation (B.10) holds exactly and $\phi(\cdot)$ plays no independent role in determining π^*.

C Exact Formula for Old–Age Consumption Inequality G_2

This appendix derives an explicit closed–form expression for

$$G_2 \equiv \frac{c_2^R}{c_2^K}$$

in the stationary steady state under log utility.

Throughout we assume $u(c) = \ln c$, the CIA constraint binds for Ricardian households, and the steady–state gross inflation rate is:

$$\pi \equiv \frac{P_{t+1}}{P_t} = \frac{1+\eta}{1+n}, \tag{C.1}$$

as shown in Appendix B.

C.1 Prices and factor incomes in steady state

Given the steady–state capital–labor ratio k, factor prices are:

$$r = \alpha k^{\alpha-1}(1-\lambda)^{\gamma_1}\lambda^{\gamma_2}, \tag{C.2}$$

$$w^R = \gamma_1 k^{\alpha}(1-\lambda)^{\gamma_1-1}\lambda^{\gamma_2}, \tag{C.3}$$

$$w^K = \gamma_2 k^{\alpha}(1-\lambda)^{\gamma_1}\lambda^{\gamma_2-1}. \tag{C.4}$$

All objects in what follows are steady–state values.

C.2 Ricardian old–age consumption c_2^R in closed form

Ricardians choose capital saving s and money balances m^R when young subject to:

$$c_1^R + s + m^R = w^R,$$

and when old

$$c_2^R = \frac{m^R}{\pi} + (1-\tau_k)rs,$$

with binding CIA

$$\frac{m^R}{\pi} = \xi c_2^R. \tag{C.5}$$

Combining the old budget constraint with Equation (C.5) gives:

$$(1-\xi)c_2^R = (1-\tau_k)rs \quad \Rightarrow \quad c_2^R = \kappa s, \qquad \kappa \equiv \frac{(1-\tau_k)r}{1-\xi}. \tag{C.6}$$

Binding CIA also implies

$$m^R = \pi\xi c_2^R = \pi\xi\kappa s. \tag{C.7}$$

Substituting Equation (C.7) into the young budget yields:

$$c_1^R = w^R - \chi s, \qquad \chi \equiv 1 + \pi\xi\kappa.$$

Under log utility, the Ricardian chooses s to maximize:

$$\ln(w^R - \chi s) + \beta_R \ln(\kappa s).$$

The first–order condition gives:

$$s = \frac{\beta_R}{1+\beta_R} \cdot \frac{w^R}{\chi} = \frac{\beta_R}{1+\beta_R} \cdot \frac{w^R}{1+\pi\xi\kappa}. \tag{C.8}$$

Combining Equations (C.6) and (C.8) yields a closed form for Ricardian old–age consumption:

$$c_2^R = \kappa s = \frac{\beta_R}{1+\beta_R} \cdot \frac{\kappa}{1+\pi\xi\kappa}\, w^R = \frac{\beta_R}{1+\beta_R} \cdot \frac{\frac{(1-\tau_k)r}{1-\xi}}{1+\pi\xi\frac{(1-\tau_k)r}{1-\xi}}\, w^R. \tag{C.9}$$

C.3 Keynesian old–age consumption c_2^K in closed form

Keynesians can only save in money. Their constraints are:

$$c_1^K + m^K = w^K, \qquad c_2^K = \frac{m^K}{\pi} + T,$$

where T is the real lump–sum transfer paid to each old Keynesian household.

Under log utility, the Keynesian chooses m^K to maximize:

$$\ln(w^K - m^K) + \beta_K \ln\left(\frac{m^K}{\pi} + T\right).$$

The first–order condition implies:

$$m^K = \frac{\beta_K w^K - \pi T}{1+\beta_K}, \tag{C.10}$$

and therefore

$$c_2^K = \frac{m^K}{\pi} + T = \frac{\beta_K}{1+\beta_K}\left(\frac{w^K}{\pi} + T\right). \tag{C.11}$$

C.4 Transfer T in closed form

The government rebates: i) capital tax revenue, and ii) real seigniorage revenue to old Keynesians.

Capital tax revenue. In period t, capital tax revenue is $\tau_k r K$. With $K = kL$ and $L = N_t = (1+n)N_{t-1}$, revenue per old Keynesian (there are λN_{t-1} old Keynesians) is:

$$T^{(k)} = \frac{\tau_k r K}{\lambda N_{t-1}} = \frac{1+n}{\lambda}\,\tau_k r k. \tag{C.12}$$

Seigniorage revenue. Real seigniorage in period t is:

$$S = \frac{M_t - M_{t-1}}{P_t}.$$

Using $M_t = (1+\eta)M_{t-1}$ gives $S = \frac{\eta}{1+\eta}\frac{M_t}{P_t}$. Moreover, aggregate real money held by the young at the end of t is:

$$\frac{M_t}{P_t} = N_t \bar{m}, \qquad \bar{m} \equiv (1-\lambda)m^R + \lambda m^K.$$

Thus

$$S = \frac{\eta}{1+\eta} N_t \bar{m} = \frac{\eta}{1+\eta}(1+n)N_{t-1}\bar{m},$$

and seigniorage per old Keynesian equals

$$T^{(s)} = \frac{S}{\lambda N_{t-1}} = \frac{1+n}{\lambda}\cdot\frac{\eta}{1+\eta}\,\bar{m}. \tag{C.13}$$

Money demand components. Binding CIA implies $m^R = \pi\xi c_2^R$ (from Equation (C.5)), hence

$$m^R = \pi\xi c_2^R. \tag{C.14}$$

Keynesian money demand is Equation (C.10). Therefore

$$\bar{m} = (1-\lambda)\pi\xi c_2^R + \lambda\frac{\beta_K w^K - \pi T}{1+\beta_K}. \tag{C.15}$$

Transfer fixed point and solution. Total transfer per old Keynesian is:

$$T = T^{(k)} + T^{(s)} = \frac{1+n}{\lambda}\left[\tau_k r k + \frac{\eta}{1+\eta}\bar{m}\right]. \tag{C.16}$$

Substituting Equation (C.15) into Equation (C.16) and solving for T yields the closed form:

$$T = \frac{\frac{1+n}{\lambda}\tau_k r k + \frac{1+n}{\lambda}\frac{\eta}{1+\eta}\left[(1-\lambda)\pi\xi c_2^R + \lambda\frac{\beta_K w^K}{1+\beta_K}\right]}{1 + (1+n)\frac{\eta}{1+\eta}\frac{\pi}{1+\beta_K}}. \tag{C.17}$$

Expression (C.17) is explicit given (k, r, w^K, π, c_2^R), and c_2^R is explicit from Equation (C.9).

C.5 Exact closed form for G_2

By definition,

$$G_2 = \frac{c_2^R}{c_2^K}.$$

Using Equation (C.11),

$$G_2 = \frac{c_2^R}{\frac{\beta_K}{1+\beta_K}\left(\frac{w^K}{\pi} + T\right)} = \frac{1+\beta_K}{\beta_K}\cdot\frac{c_2^R}{\frac{w^K}{\pi} + T}. \tag{C.18}$$

Substituting the explicit expressions (C.9) and (C.17) into (C.18) gives the exact formula:

$$G_2 = \frac{1+\beta_K}{\beta_K}\cdot\frac{\frac{\beta_R}{1+\beta_R}\cdot\frac{\frac{(1-\tau_k)r}{1-\xi}}{1+\pi\xi\frac{(1-\tau_k)r}{1-\xi}}w^R}{\frac{w^K}{\pi} + \frac{\frac{1+n}{\lambda}\tau_k r k + \frac{1+n}{\lambda}\frac{\eta}{1+\eta}\left[(1-\lambda)\pi\xi c_2^R + \lambda\frac{\beta_K w^K}{1+\beta_K}\right]}{1+(1+n)\frac{\eta}{1+\eta}\frac{\pi}{1+\beta_K}}}. \tag{C.19}$$

In Equation (C.19), c_2^R inside the numerator of the transfer term is given by Equation (C.9), and (r, w^R, w^K) are given by Equations (C.2)–(C.4). Finally, π is given by Equation (C.1). Therefore G_2 is a fully explicit function of

$$(\lambda, \gamma_1, \gamma_2, \alpha, \beta_R, \beta_K, \xi, \eta, n, \tau_k)$$

once the steady–state capital–labor ratio k is pinned down by the model's steady–state capital accumulation condition (derived in the main text).

D Derivation of the Full Hessian-Based FOC System

The Ramsey planner chooses the vector of instruments $\theta = (\eta, \tau_k, \theta_z)$ to maximize the welfare function $\mathcal{W}(\theta)$ subject to equilibrium constraints. Let k^* (capital-labor ratio) and T^* (real lump-sum transfer) be the key endogenous steady-state variables that summarize the effects of policy on efficiency and redistribution, respectively.

The optimization problem is:

$$\max_{\eta,\tau_k} \mathcal{W}(\eta, \tau_k, \theta_z)$$

where $\mathcal{W}(\eta, \tau_k, \theta_z) \equiv \mathcal{W}(k^*(\eta, \tau_k, \theta_z), T^*(\eta, \tau_k, \theta_z), \theta_z)$. We treat the third instrument θ_z as an exogenous parameter that influences the optimum (η^*, τ_k^*).

D.1 First-order conditions (FOCs)

At the interior optimum, the two FOCs with respect to η and τ_k must be satisfied:

$$\frac{d\mathcal{W}}{d\eta} = 0 \tag{D.1}$$

$$\frac{d\mathcal{W}}{d\tau_k} = 0 \tag{D.2}$$

Applying the chain rule (and suppressing the dependence of other variables on k^* and T^* for clarity, as per the typical envelope argument where the effects are dominated by the two largest equilibrium effects):

$$\frac{d\mathcal{W}}{d\eta} = \frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial k^*}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial T^*}{\partial \eta} + \left.\frac{\partial \mathcal{W}}{\partial \eta}\right|_{\text{direct}} = 0$$

$$\frac{d\mathcal{W}}{d\tau_k} = \frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial k^*}{\partial \tau_k} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial T^*}{\partial \tau_k} + \left.\frac{\partial \mathcal{W}}{\partial \tau_k}\right|_{\text{direct}} = 0$$

Note that $\left.\frac{\partial \mathcal{W}}{\partial \theta_j}\right|_{\text{direct}}$ includes effects not mediated through k^* or T^*, such as the direct effect of τ_k on capital income or η on inflation/seigniorage revenue.

D.2 Total differential of the FOCs

To find how the optimal η^* and τ_k^* change when θ_z changes, we take the total differential of the system of FOCs (Equations (D.1) and (D.2)). Let $J_1 = \frac{d\mathcal{W}}{d\eta}$ and $J_2 = \frac{d\mathcal{W}}{d\tau_k}$.

The total differential of $J_1 = 0$ is:

$$dJ_1 = \frac{\partial J_1}{\partial \eta} d\eta + \frac{\partial J_1}{\partial \tau_k} d\tau_k + \frac{\partial J_1}{\partial \theta_z} d\theta_z = 0$$

The total differential of $J_2 = 0$ is:

$$dJ_2 = \frac{\partial J_2}{\partial \eta} d\eta + \frac{\partial J_2}{\partial \tau_k} d\tau_k + \frac{\partial J_2}{\partial \theta_z} d\theta_z = 0$$

We can rewrite this as a matrix system where the left-hand side contains the unknown policy adjustments $(d\eta, d\tau_k)$ and the right-hand side contains the forcing terms due to $d\theta_z$:

$$\begin{pmatrix} \frac{\partial^2 \mathcal{W}}{\partial \eta^2} & \frac{\partial^2 \mathcal{W}}{\partial \eta \partial \tau_k} \\ \frac{\partial^2 \mathcal{W}}{\partial \tau_k \partial \eta} & \frac{\partial^2 \mathcal{W}}{\partial \tau_k^2} \end{pmatrix} \begin{pmatrix} d\eta \\ d\tau_k \end{pmatrix} = - \begin{pmatrix} \frac{\partial^2 \mathcal{W}}{\partial \eta \partial \theta_z} \\ \frac{\partial^2 \mathcal{W}}{\partial \tau_k \partial \theta_z} \end{pmatrix} d\theta_z$$

D.3 Expansion of the terms (the 2 × 3 system)

To expose the underlying complexity involving the Jacobian matrix of the equilibrium and the Hessian matrix of the welfare function, we expand the terms $\frac{\partial^2 \mathcal{W}}{\partial \theta_i \partial \theta_j}$.

The general structure of the second derivative $\frac{\partial J_1}{\partial \eta} = \frac{\partial^2 \mathcal{W}}{\partial \eta^2}$ involves three types of terms: i) *Hessian terms* (second derivatives of $\mathcal{W}$ with respect to state variables k^* and T^*); ii) *Jacobian terms* (first derivatives of state variables $\frac{\partial k^*}{\partial \eta}$ and $\frac{\partial T^*}{\partial \eta}$); and iii) *Mixed partials* (first derivatives of $\mathcal{W}$ with respect to state variables multiplied by second derivatives of state variables).

For clarity, let $J_{ij} \equiv \frac{\partial^2 \mathcal{W}}{\partial \theta_i \partial \theta_j}$. The elements of the system are:

Self-adjustment term ($J_{11} = \frac{\partial^2 \mathcal{W}}{\partial \eta^2}$):

$$J_{11} = \frac{\partial}{\partial \eta}\left(\frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial k^*}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial T^*}{\partial \eta} + \left.\frac{\partial \mathcal{W}}{\partial \eta}\right|_{\text{direct}}\right)$$

$$\begin{aligned} J_{11} = {} & \frac{\partial^2 \mathcal{W}}{\partial (k^*)^2}\left(\frac{\partial k^*}{\partial \eta}\right)^2 + 2\frac{\partial^2 \mathcal{W}}{\partial k^* \partial T^*}\left(\frac{\partial k^*}{\partial \eta}\frac{\partial T^*}{\partial \eta}\right) + \frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial^2 k^*}{\partial \eta^2} \\ & + \frac{\partial^2 \mathcal{W}}{\partial (T^*)^2}\left(\frac{\partial T^*}{\partial \eta}\right)^2 + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial^2 T^*}{\partial \eta^2} + \frac{\partial^2 \mathcal{W}}{\partial k^* \partial \eta}\frac{\partial k^*}{\partial \eta} + \frac{\partial^2 \mathcal{W}}{\partial T^* \partial \eta}\frac{\partial T^*}{\partial \eta} + \left.\frac{\partial^2 \mathcal{W}}{\partial \eta^2}\right|_{\text{direct}} \end{aligned}$$

(The derivation for $J_{22} = \frac{\partial^2 \mathcal{W}}{\partial \tau_k^2}$ is symmetric, replacing η with τ_k).

Cross-adjustment term ($J_{12} = \frac{\partial^2 \mathcal{W}}{\partial \eta \partial \tau_k}$):

$$J_{12} = \frac{\partial}{\partial \tau_k}\left(\frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial k^*}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial T^*}{\partial \eta} + \left.\frac{\partial \mathcal{W}}{\partial \eta}\right|_{\text{direct}}\right)$$

$$\begin{aligned} J_{12} = {} & \frac{\partial^2 \mathcal{W}}{\partial (k^*)^2}\left(\frac{\partial k^*}{\partial \eta}\frac{\partial k^*}{\partial \tau_k}\right) + \frac{\partial^2 \mathcal{W}}{\partial T^* \partial k^*}\left(\frac{\partial T^*}{\partial \eta}\frac{\partial k^*}{\partial \tau_k} + \frac{\partial k^*}{\partial \eta}\frac{\partial T^*}{\partial \tau_k}\right) \\ & + \frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial^2 k^*}{\partial \eta \partial \tau_k} + \frac{\partial^2 \mathcal{W}}{\partial T^* \partial \tau_k}\frac{\partial T^*}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial^2 T^*}{\partial \eta \partial \tau_k} + \left.\frac{\partial^2 \mathcal{W}}{\partial \eta \partial \tau_k}\right|_{\text{direct}} \end{aligned}$$

Forcing term ($J_{1\theta_z} = \frac{\partial^2 \mathcal{W}}{\partial \eta \partial \theta_z}$):

$$J_{1\theta_z} = \frac{\partial}{\partial \theta_z}\left(\frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial k^*}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial T^*}{\partial \eta} + \left.\frac{\partial \mathcal{W}}{\partial \eta}\right|_{\text{direct}}\right)$$

$$\begin{aligned} J_{1\theta_z} = {} & \frac{\partial^2 \mathcal{W}}{\partial (k^*)^2}\left(\frac{\partial k^*}{\partial \eta}\frac{\partial k^*}{\partial \theta_z}\right) + \frac{\partial^2 \mathcal{W}}{\partial T^* \partial k^*}\left(\frac{\partial T^*}{\partial \eta}\frac{\partial k^*}{\partial \theta_z} + \frac{\partial k^*}{\partial \eta}\frac{\partial T^*}{\partial \theta_z}\right) \\ & + \frac{\partial \mathcal{W}}{\partial k^*}\frac{\partial^2 k^*}{\partial \eta \partial \theta_z} + \frac{\partial^2 \mathcal{W}}{\partial T^* \partial \theta_z}\frac{\partial T^*}{\partial \eta} + \frac{\partial \mathcal{W}}{\partial T^*}\frac{\partial^2 T^*}{\partial \eta \partial \theta_z} + \left.\frac{\partial^2 \mathcal{W}}{\partial \eta \partial \theta_z}\right|_{\text{direct}} \end{aligned}$$

(The derivation for $J_{2\theta_z} = \frac{\partial^2 \mathcal{W}}{\partial \tau_k \partial \theta_z}$ is symmetric, replacing η with τ_k).

D.4 The full 2×3 system

The completed system describing the optimal policy adjustment $(d\eta, d\tau_k)$ in response to the perturbation $d\theta_z$ is:

$$\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} d\eta \\ d\tau_k \end{pmatrix} = -d\theta_z \begin{pmatrix} J_{1\theta_z} \\ J_{2\theta_z} \end{pmatrix}$$

where $J_{21} = J_{12}$ by the symmetry of the Hessian of $\mathcal{W}$ (assuming sufficient differentiability).

This matrix equation is the rigorous representation of the system and, as noted, involves not just the Jacobian of the state variables (first partial derivatives, $\frac{\partial k^*}{\partial \theta_j}$), but also the full Hessian matrix of the welfare function ($\frac{\partial^2 \mathcal{W}}{\partial z_i \partial z_j}$) and second derivatives of the state variables ($\frac{\partial^2 k^*}{\partial \theta_i \partial \theta_j}$).